Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2015,**35B**(4):807–831
Acta Mathematica Scientia 数学物理学报
http://actams.wipm.ac.cn

TWENTY-EIGHT YEARS WITH "HYPERBOLIC CONSERVATION LAWS WITH RELAXATION"*

Dedicated to Professor Tai-Ping Liu on the occasion of his 70th birthday

Corrado MASCIA
Dipartimento di Matematica "G. Castelnuovo" Sapienza, Università di Roma P.le Aldo Moro, 2 - 00185 Roma, ITALY
E-mail: *mascia@mat.uniroma1.it*

Abstract This paper is a review on the results inspired by the publication "Hyperbolic conservation laws with relaxation" by Tai-Ping Liu [1], with emphasis on the topic of nonlinear waves (specifically, rarefaction and shock waves). The aim is twofold: firstly, to report in details the impact of the article on the subsequent research in the area; secondly, to detect research trends which merit attention in the (near) future.

Key words conservation laws; relaxation; nonlinear waves; stability

2010 MR Subject Classification 35L65; 35B35; 01A65

In 1987, Communications in Mathematical Physics published "Hyperbolic conservation laws with relaxation" by Tai-Ping Liu, [1]. After twenty-eight years, the paper trespassed the enviable amount of three hundreds citations (as revealed by combining the results in the databases MathScinet, Scopus and World of Science). Even if no bibliometric criterium is able to assess the importance of a scientific contribution, in the case under study, such result is coherent with the many qualities which makes of [1] an unmissable classic in conservation laws. Here, the intent is to take stock of the impact of the article and to freshen the attention on the research trends pointed out by it, many of which still deserve investigation.

1 The Whys and Wherefores

Three main facts contributed to make of [1] an exemplary research paper: the relevance of the mechanism incorporated in the class of conservation laws, the (nonlinear) validation of a readable stability condition, the subsequent publication of influential complementary papers.

1.1 Relaxation Structure

"Hyperbolic conservation law with relaxations" deals with systems of the form

$$\partial_t u + \partial_x f(u,v) = 0, \qquad \partial_t v + \partial_x g(u,v) = h(u,v), \tag{1.1}$$

*Received February 3, 2015. The author is supported partially supported by the italian Project FIRB 2012 "Dispersive dynamics: Fourier Analysis and Variational Methods".

where the unknown (u, v) is a function of $(x, t) \in \mathbb{R} \times \mathbb{R}$. Hyperbolicity (and thus local well-posedness of the corresponding Cauchy problem) is guaranteed by the requirement that the jacobian $\partial(f, g)/\partial(u, v)$ has real eigenvalues. In [1], both u and v are scalars and there are two eigenvalues, $\lambda_{1,2} = \lambda_{1,2}(u, v)$ which determine the characteristic speeds of the system. For simplicity, the attention can be restricted to the strictly hyperbolic case, meaning that $\lambda_1 < \lambda_2$ for any (u, v) under consideration.

The relaxation structure is determined by the term h, which is chosen so that the dynamics leads (u, v) toward an equilibrium manifold $\mathcal{E}_*$, assumed to be the graph of a function v_*:

$$\mathcal{E}_* := \{(u, v) \,:\, h(u, v) = 0\} = \{(u, v) \,:\, v = v_*(u)\}.$$

The reduced equation, obtained from (1.1) by substituting the dynamic equation for v with the relation $v = v_*(u)$, is called equilibrium (or relaxed) equation,

$$\partial_t u + \partial_x f_*(u) = 0, \qquad \text{where} \quad f_*(u) := f\big(u, v_*(u)\big). \tag{1.2}$$

The requirement that the equilibrium manifold $\mathcal{E}_*$ is globally attracting for the underlying space independent kinetics is related to the sign of h and it is guaranteed if

$$\partial_v h(u, v) < 0. \tag{1.3}$$

A typical form for the function h is $(v_*(u) - v)/\tau$ where the relaxation time $\tau > 0$ determines the time-scale for the relaxation mechanism. Generalizations can be obtained by considering vectorial unknowns u and v, the former describing conserved quantities, the latter relaxation variables. If u is not scalar, it is crucial to determine the nature of the equilibrium equation (1.2), whose hyperbolicity is not guaranteed without specific assumptions.

Relaxation phenomena are ubiquitous in Applied Mathematics, relevant fields being the modelling of visco-elastic materials, and the study of traffic flows. Notable are the discrete-velocity reductions of the Boltzmann equation –such as the Broadwell model– which are semi-linear hyperbolic systems fitting into the family of conservation laws with relaxation.

The basic structure of relaxation models, as in the Extended Thermodynamics approach, presumes the presence of a reduced equation (or system) describing the dynamics close to an equilibrium configuration and an enlarged system dictating that, in the out-of-equilibrium regime, fluctuations are damped out by the relaxation mechanism. System (1.1) encompasses many of the basic features of such an extensive setting and, at the same time, is amenable of a rigorous approach in the hope of gaining insights valid in the general case.

1.2 Subcharacteristic Condition

Before [1], relaxation effects had already been explored in the milestone treatise [2]. Specifically, in Chapter 10, Whitham discusses, by means of relevant examples for gas-dynamics and traffic flows modelling, the role of the relation between the characteristic speeds λ_1, λ_2 of the principal part of (1.1) and the equilibrium speed

$$\lambda_* = \lambda_*(u) = f_*'(u) = \partial_u f - \partial_v f \, \partial_v h^{-1} \partial_u h\Big|_{\mathcal{E}_*} \tag{1.4}$$

of the equation (1.2), detecting the fundamental task played by the subcharacteristic condition

$$\lambda_1(u, v) < \lambda_*(u) < \lambda_2(u, v). \tag{1.5}$$

In [2], the condition is mainly motivated by means of a linear stability analysis. The extension to the nonlinear case is non-trivial since, in this situation,

> we do not normally have the luxury of complete exact solutions ([2], p.353).

"Hyperbolic conservation laws with relaxation" shows that (1.5) is relevant also in the nonlinear case (1.1), showing that the subcharacteristic condition guarantees nonlinear stability of both rarefaction and shock waves in the small amplitude regime and under the coupling condition

$$\partial_v f(u,v) \neq 0. \tag{1.6}$$

The justification is supported by means of a formal Chapman–Enskog's type expansion. To recall how this works, it is necessary to rewrite the component v as the sum of an equilibrium term plus a deviation v_1, assumed to be small with respect to v. Then, with the approximation $f(u,v) \approx f_*(u) + \partial_v f(u, v_*(u))v_1$, the first equation in (1.1) gives

$$\partial_t u + \partial_x f_*(u) + \partial_x\big(\partial_v f(u, v_*(u))v_1\big) = 0.$$

From the second equation in (1.1), it follows

$$\begin{aligned}\partial_v h(u, v_*(u))\, v_1 &\approx \partial_t v_*(u) + \partial_x g(u, v_*(u)) = v_*'(u)\partial_t u + \partial_x g(u, v_*(u))\\ &\approx -v_*'(u)\partial_x f_*(u) + \partial_x g(u, v_*(u))\end{aligned}$$

with the approximations $h(u,v) \approx \partial_v h(u, v_*(u))v_1$ and $\partial_t u \approx -\partial_x f_*(u)$. By substitution, a second order viscous type approximation of (1.1) emerges, namely

$$\partial_t u + \partial_x f_*(u) = \partial_x\big(\beta(u)\,\partial_x u\big), \tag{1.7}$$

with the diffusion function $\beta = \beta(u)$ given by[1]

$$\beta(u) = \frac{\partial_v f}{\partial_v h}\Big\{v_*'(u)\big(\lambda_*(u) - \partial_v g\big) - \partial_u g\Big\} = \frac{1}{\partial_v h}\big(\lambda_*(u) - \lambda_1\big)\big(\lambda_*(u) - \lambda_2\big),$$

where functions $\partial_v f, \partial_u g, \partial_v g, \partial_v h$ and λ_1, λ_2 are computed at $(u, v_*(u))$. Since v_1 is a multiple of $\partial_x u$, it can be seen that the above approximations are consistent in the regime

$$|\nabla^2 u| \ll |\nabla u| \ll |u|.$$

Stability is equivalent to dissipativity of (1.7) and thus, by the relaxation assumption (1.3), it is valid if and only if (1.5) holds along the equilibrium manifold $\mathcal{E}_*$.

Subcharacteristic condition is a simple and effective assumption, guaranteeing a protective wall against instabilities, which, as stated and proved in [1], holds also at the nonlinear level on the small amplitude regime. A great amount of articles gives credit to the Tai-Ping Liu's article precisely for the (nonlinear) validation of the subcharacteristic condition.

1.3 Resonance Effects

Here, "resonance" want to describe the echoing produced by the publication, a few years after [1], of other fundamental articles, which concurred in expanding the attention to hyperbolic systems with relaxation. Upon examination, four specific papers can be selected.

1993: with [3], Gui Qiang Chen and Tai-Ping Liu brought the attention on the zero relaxation limit problem, that is on giving a rigorous proof of the convergence of solutions to (1.1)

[1]Unfortunately, in [1], the expression of β in formula (2.6) is uncorrect, since it substitutes $\partial_v h$ with -1.

to the ones of the equilibrium equation (1.2). Key instruments such as entropy pairs, energy estimates and compensated compactness play a crucial role, affecting the subsequent research in this direction.

1994: the article [4], signed by the previous two authors together with David Levermore, is an unquestionable milestone in the area of conservation laws.[2] The main relevant contribution is the development of a flexible approach to deal with hyperbolic systems with relaxation for general size of the unknowns u and v. Entropy pairs, Chapman-Enskog expansions, interlaced subcharacteristic condition are the principal aspects considered in the article.

1995: Shi Jin and Zhouping Xin proposed, with [5], a new class of numerical schemes for conservation laws based on a specific choice of relaxation system, which is semi-linear. In the simplest setting, the relaxation model is a special case (1.1), given by

$$\partial_t u + \partial_x v = 0, \qquad \partial_t v + a\,\partial_x u = \frac{1}{\tau}\big(v_*(u) - v\big) \qquad\qquad (a, \tau > 0). \tag{1.8}$$

Coherently, the subcharacteristic condition (1.5) reads as

$$-\sqrt{a} < v_*'(u) < \sqrt{a}. \tag{1.9}$$

The Jin–Xin approach is flexible and can be applied to any system size and to arbitrary space dimensions. As such, it has been widely explored by experts in numerical analysis and scientific computing, broadly expanding the number of researcher interested in the understanding and development of the analysis of relaxation systems.

1996: in [6], Roberto Natalini showed that the 2×2 Jin–Xin relaxation model (1.8), under the subcharacteristic condition (1.9), enjoyes a number of special properties. These descend mainly from a comparison principle for its Riemann invariants, consequence of an underlying weakly coupled quasi-monotone structure. Such resources has been employed in [6] to analyze the zero relaxation limit in wide generality and then exploited in a number of complementary directions, to provide generalizations of the result presented in [1] for the specific system (1.8), as will be summarized in the following pages.

All together, these four articles (with a major role played by [4] and [5]), together with the hectic activity of many other researchers, drew more attention on a class of system of which (1.1) is the general 2×2 representative, vastly increasing the research activity in the area[3].

1.4 A Rough View to the Descendants

Cross-checking the databases MathScinet, Scopus and World of Science provides a list of 317 articles including [1] in the bibliography. A complete analysis would require many pages. To provide a first crude view, a coarse classification of the subjects, indicating the total number of articles in the class and the corresponding percentage, is provided by the following scheme[4].

The generic designation "numerics" collects papers whose main interest is in the development of numerical algorithms, convergence and stability analysis, performing effective and meaningful simulations. Almost all of these are motivated primarily by [5] and the reference to

[2] Among the paper by Tai-Ping Liu, this is the only one which received more citations than [1].

[3] Additional evidence to the lively activity in the area is the presence of frequent quotations to papers that did not ended with a publication. Among others, two recurrent ones are "Stability of Broadwell shocks" by A.Szepessy and Z.Xin and "Shock profiles for systems of balance laws with relaxation", by H.Freistühler and Y. Zeng. It would be interesting, at least for historical reasons, to know what led to such an incomplete end.

[4] Some articles may enter in more than one compartment.

[1] is to support the importance of the subcharacteristic condition (1.5) at the nonlinear level. The next significant slice, classified "nonlinear waves", gathers a remarkable amount of papers which developed the aspect of existence and, mainly, stability of the typical wave structure supported by conservation laws: rarefactions and shocks. This is the direction that is closer to the original spirit and content of [1] and, for this reason, will now be explored in details.

topic	$n.$	%	topic	$n.$	%
numerics	98	31	nonlinear waves	84	27
zero relaxation limit	50	16	specific models	49	15
initial value problems	44	14	multi-d analysis	17	5
stability conditions	16	5	all	317	100

2 Diffusion Waves and Rarefactions

The parabolic approximation (1.7) is deduced by means of a Chapman–Enskog expansion and, as such, it is valid for solutions with derivatives which are smaller than the solution itself. Such regime is valid for small perturbation of constant states and of expansive solutions, corresponding, respectively, to diffusion and rarefaction waves.

2.1 Diffusion Waves

Diffusion waves are quickly discussed in [1] and their stability is supported by means of the formal time-asymptotic derivation of the viscous approximation (1.7). Tai-Ping Liu states that

> for a perturbation of an equilibrium state, (1.7) governs the dissipation process as $t \to +\infty$; the justification would involve the study of linearization of (1.1) around the nonlinear diffusion waves provided by (1.7) ([1], p.157).

A rigorous account of this approach is given in [7] and it is briefly described here.

Let $\nu > 0$ and f_* be a given smooth function. The diffusion wave $\theta = \theta(x,t)$ carrying mass m is the solution to the Cauchy problem

$$\partial_t u + \partial_x \Big\{ f_*'(0)\, u + \frac{1}{2}\, f_*''(0)\, u^2 \Big\} = \nu\, \partial_{xx} u, \qquad u(x,-1) = m\, \delta(x), \tag{2.1}$$

where δ is the Dirac mass concentrated at 0. Thanks to the Hopf–Cole transformation, such solution can be written explicitly in term of error functions. Moreover, it can be shown that θ describes the large-time behavior of small perturbations of a constant state (chosen to be 0) for the equation in (2.1). The goal is to show that an analogous description holds for (1.1).

For initial datum $u_0 \in L^1$ and with $m = \int_{\mathbb{R}} u_0$, Chern decomposes (u, v) as

$$u = \theta + \partial_x z, \qquad v = v_*(\theta) - \frac{\partial_x \theta}{\partial_v f} + w$$

transforming the problem into the analysis of the evolution of the couple (z, w). The system for the new unknown can be explored by combining Fourier method for the linearized equation and energy estimates. As a result, local attractivity of diffusion waves for (1.1) is established.

Theorem 2.1 (I.-L. Chern 1995 [7]) Assume the subcharacteristic condition (1.5) at $(0, v_*(0))$. Let the mass $m = \int_{\mathbb{R}} u_0$ and the initial data $(z_0, w_0) \in (L^1 \cap H^3) \times (L^1 \cap H^2)$ be sufficiently small. Then, there holds

$$\begin{aligned} &|u(\cdot,t)-\theta(\cdot,t)|_{L^p} \le C(1+t)^{-1+1/2p} && 1\le p\le\infty,\\ &|v(\cdot,t)-v_*\big(\theta(\cdot,t)\big)|_{L^p} \le C(1+t)^{-1+1/2p} && 2\le p\le\infty. \end{aligned}$$

The decay rates should be regarded as the ones for the $L^1 \to L^p$ case. As such, the distance from the diffusion wave θ decayes with an exponent that is bigger by 1/2 with respect to the absolute decay of the perturbation, as for the space derivative for the heat equation. The presence of such precise algebraic decay (with a gain of 1/2) is the consequence of an additional space decay assumed on the initial datum (see discussion in [8], Section 5).

The special structure of the Jin–Xin system (1.8) permits to prove more general results. In [8], for the case of $v_*(u) = \alpha\, u^2/2$, H. Liu and R. Natalini proved that convergence to diffusion waves holds for initial data $(u_0, v_0) \in L^1 \times L^1$ such that

$$|v_0| \le \sqrt{a}\, u_0 \qquad \sqrt{a}\, u_0' \pm v_0' \le C$$

for some constant $C > 0$. The argument is based on a rescaling approach, a method previously used to describe asymptotic behavior of parabolic equation with power-like nonlinear dependencies, and it requires no smallness assumption (execpt for the above pointwise estimate) on the initial perturbation and on the mass m. Because of the few requirements on the initial datum, the convergence is weaker with respect to the one given in Theorem 2.1: there holds

$$\lim_{t\to+\infty} t^{\frac12\left(1-\frac1p\right)} |u(\cdot,t)-\theta(\cdot,t)|_{L^p} = 0 \qquad \forall\, p\in[1,+\infty).$$

(see [8], Theorem 1.1). Anyway, this is enough to state that convergence of u to the diffusion wave θ is faster than the decay of the diffusion wave itself.

The Jin–Xin system can be reduced to a one-field equation for u,

$$\partial_t u + \partial_x v_*(u) = \tau\big(a\,\partial_{xx} u - \partial_{tt} u\big).$$

For general initial data $(u, \partial_t u)\big|_{t=0} = (u_0, u_1)$, the mass $m(t) := \int_{\mathbb{R}} u\,\mathrm{d}x$ is not conserved, being

$$m(t) = m_0 + \tau(1-\mathrm{e}^{-t/\tau})\, m_1 \qquad \text{where} \quad m_0 := \int_{\mathbb{R}} u_0(x)\mathrm{d}x, \quad m_1 := \int_{\mathbb{R}} u_1(x)\mathrm{d}x.$$

Convergence to diffusion waves can still be proved, considering as mass the asymptotic value $m_0 + \tau\, m_1$. This is shown in [9], under weak assumptions on the equilibrium flux v_*, using a method based on the analysis of the linearized equation and Fourier transform, as in [7]. Initial data (u_0, u_1) are assumed to be in $(L^1_1 \cap W^{1,p}) \times (L^1_1 \cap L^p)$ where L^1_1 is the weighted space $\{u \in L^1 : (1+|x|)u \in L^1\}$. Thanks to the extra decay assumption, decay rates similar to the ones of Chern are obtained (see [9], Theorem 2.1).

An interesting variation of the Jin–Xin system is the p-system with relaxation

$$\partial_t u + \partial_x v = 0, \qquad \partial_t v + \partial_x p(u) = v_*(u) - v, \tag{2.2}$$

which is often considered as a starting test-case for quasilinear relaxation systems. The Chapman–Enskog expansion furnishes the approximation (1.7) with $\beta(u) = p'(u) - v_*'(u)^2$. Therefore, the subcharacteristic condition (1.5) reads as

$$-\sqrt{-p'(u)} < v_*'(u) < \sqrt{-p'(u)}. \tag{2.3}$$

Coherently, in [10] it is shown that the large time-behavior of an appropriate class of perturbations of the constant state $\big(u, v_*(\bar{u})\big)$ is described by a diffusion wave solution to

$$\partial_t u + f_*'(0)\,\partial_x u = \big(p'(u) - v_*'(u)^2\big)\partial_{xx} u.$$

(see [10], Theorem 2.2, p.500). The proof is based on the one-field representation for (2.2), which consists in a nonlinear wave-like equation, and on an appropriate combination of energy estimates together with information deduced by looking at the one-field equation as a perturbation of the heat equation.

Renouncing to the target of a sharp description of the large time representation of perturbation of a constant state, it can be directly analyzed the problem of proving dissipation of perturbations (with a decay rate, if possible). Stability of constant states has been considered in [11] in the case of general relaxation model endowed with a convex entropy. The approach is based on Lyapunov functionals and it is not able to furnish a detailed description of the asymptotics based on diffusion waves. The same problem has been tackled by Fourier transform and energy estimates in [12]. The method is flexible and the result is general, being also capable to provide a detailed description of the asymptotics based on the Chapman-Enskog expansion also in the multi-dimensional framework. In the same area, but limited to the Jin–Xin model in several dimensions, the validity of a weak Huygens principle has been shown in [13].

Finally, let me also quote the description of the asymptotic behavior in the case of presence of both relaxation and diffusion considered in [14] for the specific 2×2 semilinear system

$$\partial_t u + \partial_x u = \partial_{xx} u + k(v - u^q), \qquad \partial_t v = k(u^q - v), \qquad (k, q > 0).$$

The authors state that the large-time behavior is different with respect to the one of its inviscid parallel. Investigating such differences in a wider generality is a fascinating research direction.

2.2 Rarefaction Waves

If the flux f_* is convex, the solution r to the Riemann problem for (1.2) determined by

$$u(x, 0) = u_-\,\chi_{(-\infty,0)}(x) + u_+\,\chi_{(0,+\infty)}(x) \tag{2.4}$$

with $u_- < u_+$, is called rarefaction wave (here χ_I denotes the indicator function of the set I). Such solution exhibits a type of stability, in the sense that it describes the asymptotic behavior also in the case of more general initial data as far as the space-asymptotic states $u_\pm$ are preserved. Is this structure (and its stability) preserved for (1.1)?

Rarefactions have a weaker form of stability with respect to shocks, the key difference being the fact that the former are expansive solution and the latter are compressive. This induces the presence of a scarser amount of results on rarefactions with respect to the shock case[5].

The conservation law (1.2) is invariant under the dilation $(x, t) \mapsto \alpha(x, t)$. As a consequence, rarefaction waves r are self-similar solutions depending solely on the scalar quantity x/t. The dilations invariance does not hold for (1.1) and the solution of the corresponding Riemann problems is far from explicit. Actually, it couples in a non-trivial way the structure of the solutions of the principal part (defined by f and g) and the one of the equilibrium equation (1.2). Thus, the first obstacle in proving stability of rarefactions for (1.1) is that there is no evidence of what a rarefaction wave is and the choice of the definition can be crucial to complete

[5] In the case of descendants of [1], the proportion between articles on rarefactions and on shocks is about 1:2.

the analysis. The approach in [1] consists in considering a solution $\phi^{\rm Liu}$ to (1.2) with a non-increasing initial datum $\phi_0^{\rm Liu}$, coinciding with (2.4) outside the interval $[-k,k]$, and definining (relaxation) rarefaction wave the couple $\Phi^{\rm Liu} := \big(\phi^{\rm Liu},\psi^{\rm Liu}\big) = \big(\phi^{\rm Liu}, v_*(\phi^{\rm Liu})\big)$. While such couple verifies the first equation in (1.1), it does not satisfies the second: precisely,

$$\partial_t\phi^{\rm Liu}+\partial_x f(\phi^{\rm Liu},\psi^{\rm Liu})=0,\qquad \partial_t\psi^{\rm Liu}+\partial_x g(\phi^{\rm Liu},\psi^{\rm Liu})=h(\phi^{\rm Liu},\psi^{\rm Liu})+F,$$

with the error term F given by

$$F=\big\{\partial_u g+(\partial_v g-\partial_u f)v_*'-\partial_v(v_*')^2\big\}\partial_x\phi^{\rm Liu}.$$

Since the function $\partial_x\phi^{\rm Liu}$ decayes to zero as $t\to+\infty$ (with pointwise rate $1/t$), $\Phi^{\rm Liu}$ is an asymptotic exact solution to (1.1). The analysis of its nonlinear stability consists in studying the evolution of the perturbation

$$Z=(z,w):=U-\Phi^{\rm Liu}=(u,v)-(\phi^{\rm Liu},\psi^{\rm Liu}),$$

as described by the non-autonomous system

$$\begin{cases}\partial_t z+\partial_x\big\{f(\Phi^{\rm Liu}+Z)-f(\Phi^{\rm Liu})\big\}=0,\\ \partial_t w+\partial_x\big\{g(\Phi^{\rm Liu}+Z)-g(\Phi^{\rm Liu})\big\}=h(\Phi^{\rm Liu}+Z)-h(\Phi^{\rm Liu})-F,\end{cases}$$

with initial condition $Z(x,0)=Z_0$. Being the perturbation small, the equation for Z can be rewritten in vectorial form as

$$\partial_t Z+A(\phi)\partial_x Z+\big\{B(\phi)+\partial_x A(\phi)\big\}Z=\begin{pmatrix}0\\ F\end{pmatrix}+\text{H.O.T.},\tag{2.5}$$

where H.O.T. indicates high order terms and

$$A(\phi):=\frac{\partial(f,g)}{\partial(u,v)}\Big|_\Phi\quad\text{and}\quad B(\phi):=\frac{\partial(0,h)}{\partial(u,v)}\Big|_\Phi.$$

In term of the first component z of the perturbation Z, equation (2.5) translates into a nonlinear wave equation exhibiting as second order terms the principal part of the hyperbolic 2×2 system and as first order term the linearization of the limiting equation (1.2)

$$\partial_{tt}z+(\lambda_1+\lambda_2)\partial_{tx}z+\lambda_1\lambda_2\partial_{xx}z-\partial_v h\big\{\partial_t z+\partial_x(\lambda_* z)\big\}=\cdots\tag{2.6}$$

System (2.5) (or its scalar version (2.6)) is amenable to energy estimates. Depending on specific calculations or tricks, taking advantage from the assumptions on the system under consideration, different kind of results can be proved, the main differences being the size $|u_+-u_-|$ of the rarefaction wave, the size of the perturbation $Z=(z,w)$, the regularity framework.

Theorem 2.2 (T.-P. Liu 1987 [1]) Let f_* be convex and assume subcharacteristic condition (1.5) on $\mathcal{E}_*$ for $u\in[u_-,u_+]$. Then, there exists $\varepsilon_0>0$ such that if $|u_+-u_-|+|Z_0|_{W^{3,\infty}}\le\varepsilon_0$, the Cauchy problem for the relaxation system (1.1) has a global solution $U=(u,v)$ and the perturbation Z converges to 0 as $t\to+\infty$ in $L^\infty(\mathbb{R})$.

The crucial point is to establish the energy estimate, formula (3.21) in [1],

$$|Z|_{H^3}^2+\int_0^t\Big\{|\sqrt{\partial_x\lambda_*}\,Z|_{L^2}^2+|\partial_x Z|_{H^2}^2\Big\}{\rm d}s\le C\Big\{|\partial_x Z_0|_{H^2}^2+\int_0^t\int_{\mathbb{R}}|\partial_x\lambda_*|^3{\rm d}x{\rm d}s\Big\}.\tag{2.7}$$

This bound is obtained by controling all of the spurious terms emerging from the higher order part, using only the subcharacterestic condition and taking advantage of the smallness of both the rarefaction wave and the initial perturbation. Such strong assumptions, together with the high-regularity requirement (the initial perturbation is assumed to have three uniformly bounded derivatives), are counter-balanced by the wide generality of the relaxation system, supporting the statement (at the core of the paper) that the subcharacteristic condition is sufficient for local stability of nonlinear waves. At the date there is no new contribution at the same level of generality. Refinements concern with special case of relaxation system and precisely the Jin–Xin model (1.8) and the p-system with relaxation (2.2).

Concerning the Jin–Xin model, the first specific article is [15]. Here, a different version of relaxation rarefaction wave is considered, following the approach proposed by A. Matsumura and K. Nishihara studying stability of rarefaction waves for a compressible gas model with real viscosity (see [16]). Specifically, the perturbed Riemann initial datum ϕ_0^{Liu} is substituted with a smoothed version of the jump from u_- to u_+, given explicitly by

$$\phi_0^{\text{MN}}(x) = \frac{1}{2}(u_+ + u_-) + \frac{1}{\pi}(u_+ - u_-)\arctan x.$$

The main difference stems in slowly decaying tails to $u_\pm$ at the place of the compact perturbation of the pure jump, as previously considered. Correspondingly, the perturbation is

$$Z = (z, w) := U - \Phi^{\text{MN}} = (u, v) - (\phi^{\text{MN}}, \psi^{\text{MN}}),$$

where $\psi^{\text{MN}} := v_*(\phi^{\text{MN}})$.

Theorem 2.3 (H. Zhao 2000 [15]) Let f_* be convex and assume subcharacteristic condition (1.9). Then, for any $Z_0 \in H^2(\mathbb{R})$, the Cauchy problem for the Jin–Xin system (1.1) has a global solution $U = (u, v)$ and the perturbation Z converges to 0 as $t \to +\infty$ in $W^{1,\infty}(\mathbb{R})$.

The strategy is again based on the energy method. To compare with (2.7), the estimate derived in [15] is

$$\begin{aligned} &|Z|_{H^2}^2 + |\partial_t Z|_{H^1}^2 + |\partial_{tt} Z|_{L^2}^2 + \int_0^t \left\{ |\sqrt{\partial_x \phi^{\text{MN}}}\, z|_{L^2}^2 + |\partial_x Z|_{H^1}^2 + |\partial_t Z|_{H^1}^2 + |\partial_{tt} Z|_{L^2}^2 \right\} \mathrm{d}s \\ &\le C\left\{ |Z_0|_{H^2}^2 + 1 \right\}. \end{aligned}$$

Apart for some technical details relative to the functional space framework, it is remarkable that no assumption on the size of the wave and of the initial perturbation is required. As such, the result is very satisfactory and much more general than the original by T.-P. Liu, but it is heavily based on special qualities of the Jin–Xin model.

The subsequent contribution [17] has a number of strenghts, based on a deeper use of the properties satisfied by the Jin–Xin system (see [6]). The main statement requires a minimum of regularity, no smallness assumptions on both wave/perturbation size and, in addition, gives also decay rates for the first time in the case of relaxation models. The definition of relaxation rarefaction wave is different with respect to both the previous ones. Precisely, it is considered as rarefaction the couple $\Phi^{\text{HL}} = (\phi^{\text{HL}}, \psi^{\text{HL}})$, solution to (1.8) with initial datum

$$u(x, 0) = r(x, t_0), \qquad v(x, 0) = v_*\big(r(x, t_0)\big),$$

for some t_0 sufficiently large, where r indicates here the solution to (1.2) with initial datum (2.4). Such solution has the advantage of being an exact solution to the relaxation system, differently with respect to Φ^{Liu} and Φ^{MN}. Estimates on Φ^{HL} descend from the characteristic method and the maximum principle, which can be used only for the Jin–Xin case.

To state the result, let us set $Z = U - \Phi^{\rm HL}$ with initial datum Z_0.

Theorem 2.4 (H. Liu 2001 [17]) Let f_* be convex and assume subcharacteristic condition (1.9). Then, for any $Z_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$, the Cauchy problem for the Jin–Xin system (1.1) has a global solution $U = (u, v)$ and the perturbation Z satisfies the bound

$$|U - \Phi^{\rm HL}|_{L^p} \le C(1+t)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}.$$

The argument is based on the L^1-contraction property of the Jin–Xin model and on two energy estimate. The first has a spirit similar to the previous ones

$$|z|^2_{H^1} + |\partial_x w|^2_{L^2} + \int_0^t |\partial_x Z|^2_{L^2} \mathrm{d}s \le C\left\{|z_0|^2_{H^1} + |\partial_x w_0|^2_{L^2}\right\}.$$

The second is time weighted and it is a key ingredient in determining the decay rates

$$(1+t)^\gamma \left\{|z|^2_{H^1} + |\partial_x w|^2_{L^2}\right\} + \int_0^t (1+t)^\gamma |\partial_x Z|^2_{L^2} \mathrm{d}s \le C(1+t)^{\gamma - 1/2}.$$

Theorem 2.4 is not directly comparable with Theorems 2.2 and 2.3 since the rarefaction wave takes values outside the equilibrium manifold $\mathcal{E}_*$ for $t > 0$ and the distance between the trajectory and the manifold is not explicitly estimated. A preliminary attempt is presented in the final Section of [17], where a bound on the distance of the relaxation rarefaction wave from the relaxed rarefaction wave is given, but, as noted by the author, is not optimal. The conjecture stated at the very end, formula (4.2) of the reference, is still open.

Later on, a similar approach has been used in [18] considering as a relaxation rarefaction wave $\Phi^{\rm MN}$. As a consequence, the perturbation equation exhibits a source term (absent in the H.Liu's approach) and the final result gives a decay rate of the L^p-norm of the form $(1+t)^{-(1-2/p)/3}$, which does not seem to be optimal.

The latest result on the Jin–Xin case is [19], with emphasis on initial data (u_0, v_0) not asymptotically at equilibrium, that is such that $(u_0, v_0)(x) \to (u_\pm, v_\pm)$ as $x \to \pm\infty$ with $v_\pm \ne v_*(u_\pm)$. As a consequence, the asymptotic values of the solution (u, v) moves toward the equilibrium manifold as time increases and an additional corrector term has to be introduced to follow the dynamic at $x = \pm\infty$. The technique uses characteristic method and piecewise energy estimates, a method capable to keep into account the behavior of the solution along the characteristics of the Jin–Xin system (see [19], Theorem 2.1).

The case of rarefactions for the p-system with relaxation (2.2) is discussed in [20] and in [21]. Both articles are dedicated to initial data with out-of-equilibrium asymptotic states, so that, as for the semilinear case, the strategy consists in decomposing the solution as

$$u(x,t) = \phi^{\rm MN}(x,t) + \hat{u}(x,t) + z(x,t), \qquad v(x,t) = \psi^{\rm MN}(x,t) + \hat{v}(x,t) + w(x,t).$$

with $\hat{u}, \hat{v}$ is a corrector for the behavior at $x = \pm\infty$. Then, C. Zhu applies a standard energy method obtaining a statement limited to small-amplitude rarefaction waves, [20]; H. Zhao and Y. Zhao remove this requirement, assuming existence of a global smooth solution, [21].

Theorem 2.5 (H. Zhao, Y. Zhao 2003 [21]) Assume that the Cauchy problem, for datum $(z_0, w_0) \in H^2$ has a global smooth solution bounded in C^1 with sufficiently small first order derivatives. If the subcharacteristic condition (2.3) is satisfied, then

$$\lim_{t \to +\infty} |\left(u - \phi^{\rm MN}, v - \psi^{\rm MN}\right)|_{W^{1,\infty}} = 0.$$

The apparent absence of smallness of the initial datum is hided in the (relevant) global existence assumption. Sufficient conditions are provided in [22] showing, as an example, that the hypothesis is satisfied for a power-law p with exponent in $[1,3]$, smallness of the C^1-norm of the initial datum and smallness of $|v_*(u)-v_*(\bar{u})|$.

Investigation on stability of rarefaction waves for systems with relaxation in several dimensions is limited to the Jin–Xin approximation of multi-dimensional scalar conservation laws. To simplify the presentation, the discussion is usually limited to the case of bi-dimensional space variable. Then, the Jin–Xin relaxation approximation of the scalar conservation law

$$\partial_t u + \partial_x f_*(u) + \partial_y g_*(u) = 0$$

is the 3×3 semilinear hyperbolic system

$$\partial_t u + \partial_x v_1 + \partial_y v_2 = 0, \quad \partial_t v_1 + a_1\,\partial_x u = f_*(u) - v_1, \quad \partial_t v_2 + a_2\,\partial_y u = g_*(u) - v_2,$$

and the subcharacteristic condition becomes $\sup\big\{(f')^2/a_1 + (g')^2/a_2\big\} < 1$.

The first stability result has been established by T. Luo in [23] and concerns with small perturbations of small rarefaction waves; the proof is based on the stability of rarefactions as proved in [1] for the one-dimensional case and on energy estimates. As previously reported, the smallness assumption in the stability of one-dimensional rarefaction waves for the Jin–Xin model has been removed in [15]. Based on that, in the same article, the result by T.Luo is extended to the multi-dimensional case. More than that, it is shown that the smallness assumption on the initial perturbation can be removed, if the functions f and g satisfy the growth assumption

$$|f^{(i)}(u)| + |g^{(i)}(u)| \le O(1)\big(1+|u|^p\big) \qquad\qquad i=1,2,3,4, \quad p\in(0,1).$$

Further improvements with respect to the size of the initial perturbations are given in [24], where it is also shown that global attractivity holds when functions f and g have either second derivative or third derivative globally bounded.

Stability results of rarefaction waves for specific $n\times n$ relaxation systems of dimension with $n>2$ are presented in [25, 26] (viscoelasticity models), [27–29] (4×4 Jin-Xin system with a 2×2 limit) [30] (shallow water wave equations), [31] (general discussion with application to flow of steam, nitrogen and water in porous media).

3 Relaxation Shocks

Differently to the rarefactions, under general assumption, relaxation system (1.1) supports exact traveling wave solution which corresponds to shock wave of (1.2). Existence amounts in analyzing a system of ordinary differential equations for the profile of the wave. Concerning stability, as previously recalled, the crucial difference with respect to rarefactions reposes in the fact that the latter are expansive, while shocks are compressive and, as such, more stable.

Comparing with the stability analysis for the equilibrium equation (1.2), a macroscopic dissimilarity resides in the fact that, in many circumstances, relaxation shocks are smooth solutions to (1.1). As such, small perturbations generate smooth solutions and techniques such as energy method and linearization can be implemented to tackle the problem. In this respect, the situation is closer to the case of viscous conservation laws with viscosity as inspired by the approximation (1.7). Anyway, it should be stressed that for relaxation shocks, the assumption on the relative size of the solution and its derivatives required for the Chapman–Enskog expansion are not verified.

3.1 Existence

Equation (1.2) supports specific travelling waves known as shock waves. These are solutions $u(x,t)=\phi(x-\sigma t)$ with profile ϕ which is piecewise continuous:

$$\phi(\xi)=u_-\,\chi_{(-\infty,0)}+u_+\,\chi_{(0,+\infty)} \tag{3.1}$$

for states $u_\pm\in\mathbb{R}$ and speed $\sigma\in\mathbb{R}$ (χ_I denotes the indicator function of the set I). The function u is a weak solution if the values σ and $u_\pm$ satisfy the Rankine–Hugoniot condition

$$\sigma\,(u_+-u_-)=f_*(u_+)-f(u_-). \tag{3.2}$$

In addition, to fit into the entropy solutions framework, the triple (u_-,u_+,σ) is required to satisfy the admissibility condition

$$\sigma<\frac{f_*(u)-f_*(u_-)}{u-u_-}\qquad\forall\,u\in\{\theta\,u_-+(1-\theta)u_+\,:\,\theta\in(0,1)\}. \tag{3.3}$$

Are shock waves persistent when considering the relaxation version (1.1) of the equilibrium equation (1.2)? A traveling wave $(u,v)(x,t)=(\phi,\psi)(x-\sigma\,t)$ to (1.1) such that

$$(\phi,\psi)(\pm\infty)=(u_\pm,v_\pm)\qquad\text{with}\quad v_\pm=v_*(u_\pm)$$

is a relaxation shock. The problem of existence of relaxation shocks is analogous to the one considered for parabolic variations of (1.2), obtained by taking into account viscous terms.

Proving existence amounts in determining the presence of heteroclinic orbits for the system

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\big\{f(\phi,\psi)-\sigma\phi\big\}=0,\qquad\frac{\mathrm{d}}{\mathrm{d}\xi}\big\{g(\phi,\psi)-\sigma\psi\big\}=h(\phi,\psi). \tag{3.4}$$

By integrating the first equation with respect to $\xi\in\mathbb{R}$ and taking into account the asymptotic conditions, the Rankine–Hugoniot condition (3.2) follows. In particular, the asymptotic states $u_\pm$ determine the speed σ, with the same relation settled for (1.2).

Integrating the first equation (3.4), there follows

$$f(\phi,\psi)-f(u_\pm,v_\pm)-\sigma(\phi-u_\pm)=0,$$

which defines implicitly a relation of the form $\psi=\Psi(\phi)$ (thanks to the coupling condition (1.6)) and reduces the system (3.4) to the scalar equation

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\big\{g\big(\phi,\Psi(\phi)\big)-\sigma\Psi(\phi)\big\}=h\big(\phi,\Psi(\phi)\big). \tag{3.5}$$

This equality encodes all the information on the existence of relaxation shocks for (1.1).

Theorem 3.1 (T.-P. Liu 1987 [1]) Let (u_-,u_+,σ) be such that (3.2) holds. Then, if the subcharacteristic condition

$$\lambda_1(\phi,\Psi(\phi))<\sigma<\lambda_2(\phi,\Psi(\phi))\qquad\forall\,\phi\in\{\theta\,u_-+(1-\theta)u_+\,:\,\theta\in[0,1]\} \tag{3.6}$$

is satisfied, the admissibility condition (3.3) is satisfied if and only if the system (1.1) has a (smooth) relaxation shock with asymptotic states $u_\pm$.

The proof can be sketched as follows. Changing the reference frame to the one moving with speed σ and assuming, without loss of generality, $f(u_\pm, v_\pm) = 0$, the function Ψ is such that $f(\phi, \Psi(\phi)) = 0$ and thus $\Psi' = -\partial_u f/\partial_v f$. Then, the subcharacteristic condition gives $\lambda_1\lambda_2 < 0$ and equation (3.5) becomes

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = F(\phi) := -\frac{h\,\partial_v f}{\lambda_1\,\lambda_2}(\phi, \Psi(\phi)). \tag{3.7}$$

The right-hand side is zero if and only if $h = 0$ and thus, starting from a value between u_- and u_+, the solution converges to two equilibria at $\pm\infty$. Such states coincide with $u_\pm$ if and only if the function h does not change sign along the graph Ψ for ϕ between u_- and u_+.

Assuming, for simplicity, the non-degeneracy condition $\sigma \neq f_*'(u_\pm)$, such sign can be determined by computing the derivative of $F = F(\phi)$ at u_-. Since, at $u_\pm$, there holds

$$F' = -\frac{\partial_v f}{\lambda_1\,\lambda_2}\left(\partial_u h + \partial_v h\Psi'\right) = \frac{1}{\lambda_1\,\lambda_2}\left(\partial_u f\partial_v h - \partial_v f\partial_u h'\right) = \frac{\partial_v h}{\lambda_1\,\lambda_2} f_*',$$

the function F has the same sign of f_* between u_- and u_+, proving the stated equivalence.

The subcharacteristic condition (3.6) differs from its original version (1.5), since it is now a requirement on the wave speed. As observed in [1], for weak traveling waves the speed is close to λ_* and thus (1.5) implies (3.6). When the amplitude is large, the speed can be super-characteristic and, in this case, the profile would contain discontinuities, referred to as sub-shocks. This makes of strong relaxation waves with jumps an intriguing issue to be investigated.

Minor variations of Theorem 3.1 has been considered in [20, 32–34] for specific 2×2 models (Jin-Xin, p-system with relaxation) with the aim of providing a sound basis before attacking the stability analysis, specifically for non-convex fluxes f_* (discussed in the next subsection).

For larger specific relaxation system, similar reduction to a scalar ODE can be performed in the case of a scalar variable v. A significant example is the Broadwell model (see [35, 36]),

$$\partial_t u + \partial_x v = 0, \quad \partial_t v + \partial_x w = 0, \quad \partial_t w + \partial_x v = \frac{1}{8\tau}\big\{(u-w)^2 - 4(w^2 - v^2)\big\}, \tag{3.8}$$

for which relaxation shocks have an explicit representation. Similarly, for the system

$$\partial_t u + \partial_x v = 0, \quad \partial_t v + \partial_x w = 0, \quad \partial_t w + E\,\partial_x v = \frac{1}{\tau}\big\{w_*(u) - w\big\} \qquad (E > 0), \tag{3.9}$$

proposed for the description of isothermal motion of a viscoelastic material (see [37] and reference therein), taking advantage of the conservative form of the first two equations, the traveling wave equation can be reduced to a scalar differential equation, amenable of a complete qualitative analysis under general assumptions and of an explicit solutions in the case of quadratic stress-strain function w_* as computed in [38].

Also the model for reactive flows considered in [39] is 3×3 with two conserved quantities, but it is complicated by its quasilinear structure. The existence theorem proved by R. Pan is limited to the small-amplitude case and a detailed description of the large-amplitude case (with attention to the eventual subshocks) has apparently not yet been performed. Other simple examples can be found in [40] (barotropic Euler equation with relaxation, 3×3) and in [41] (Euler equations for a gas in non-thermal equilibrium, 4×4).

In [42], relaxation shocks for the Kerr–Debye system, a model for electromagnetic wave propagation in nonlinear media, are analyzed. In one space dimension[6], the system reduces to

$$\partial_t u + \partial_x v = 0, \quad \partial_t v + \partial_x\Big(\frac{u}{1+w}\Big) = 0, \quad \partial_t w = \frac{1}{\tau}\Big\{\frac{u^2}{(1+w)^2} - w\Big\}. \tag{3.10}$$

The existence result is pretty much in the spirit of Theorem 3.1 and it is worthwhile to be presented, leaving to the reader the task of adapting the terminology to the present case.

Theorem 3.2 (D. Aregba-Driollet and B. Hanouzet 2011 [42]) Let the asymptotic states $(u_\pm, v_\pm, w_\pm)$ and the speed σ be such that the Rankine–Hugoniot conditions hold. Then, a relaxation shock for (3.10) exists if and only if $u_- \neq u_+$, $u_- \cdot u_+ > 0$ and the asymptotic states $(u_\pm, v_\pm, w_\pm)$ are connected by an admissible shock wave for the corresponding relaxed system.

When the size of the relaxing component v increases, the profile ODE has no scalar reduction and the analysis is much more intricated. The available results for the large amplitude case are few and sparse. It is worthwhile to mention the discussions on the Grad 13-moments model, a 5×5 system with three conserved quantities, which is nowadays a classic in Extended Thermodynamics. The profile equation is two-dimensional and existence of relaxation shocks is a matter of finding a heteroclinic connection in the plane. No complete result in the sense of Theorem 3.1 is available for such model. Interesting discussions can be found in [43] (with specific considerations on the presence of subshocks) and in [44] (where relaxation shocks are analyzed by using the dynamical systems approach and, in particular, local bifurcation analysis).

When the attention is restricted to small amplitude relaxation shocks, the existence problem can be solved for a broad class of systems. Entering in the details of the statement would require a minute discussion on the structure of the relaxation system in the general case; hence, for the sake of conciseness, the subsequent presentation is limited to a shortly commented list of the key references. W.-A. Yong and K. Zumbrun were the first to show that a statement analogous to the one for small-amplitude viscous shocks holds also for relaxation shocks, [45]. A modification of the proof was presented in [46], to show that the statement is valid also without the "genuine coupling condition" of [45]. A further step, consisting in eliminating the non-characteristic shock speed assumption, present in both the two quoted papers, has been proposed in [47] (using an expansion center-manifold near the degenerate point), in [48] (valid for semilinear system and based on a micro-macro decomposition in the Boltzmann equation' style) and in [49] (extending to the latter strategy to the quasilinear case, by a delicate application of Nash-Moser iteration). The strategy of the last two references, ineherently limited to the small-amplitude case, has the advantage of possible adaptation to the infinite-dimensional case, as required by the Boltzmann equations, as shown in [50]. The analysis of shocks for the Boltzmann equation is an intriguing research trend to which Tai-Ping Liu himself participates actively (see [51] and references therein). The shock wave problem for discrete velocity model with an arbitrary finite number of velocities has been also addressed in [52] by means of dynamical systems' techniques.

Compresence of relaxation and viscosity has also been considered, motivated by the surprising observation in [53] that when both the smoothing factors are present, dispersive waves may be enhanced. In [54], it is shown that for a p-system with relaxation and real viscosity,

$$\partial_t u + \partial_x v = 0, \qquad \partial_t v + \partial_x p(u) = \frac{1}{\tau}\big\{v - v_*(u)\big\} + \mu\partial_{xx} v$$

[6] For space/time reason, the multidimensional analysis proposed in [42] is not discussed here.

traveling waves exist if the subcharacteristic condition holds and the diffusion coefficient is sufficiently small with respect to the relaxation time. Adding viscosity, the ODE system for the traveling waves has order three that can be reduced to two, since the first equation is conservative. Thus, differently with respect to the case (1.1), the equation is not scalar and requires a more delicate phase-plane analysis. A careful application of singular perturbation's techniques permits to show in [55] that the viscous-relaxation profiles converge to the corresponding relaxation profiles as the viscosity coefficient μ tends to zero. Such property is relevant to prove a nonlinear stability results (discussed in the next subsection). I am not aware of any attempt of discussing viscous modifications of the relaxation system (1.1) in its general form.

Finally, let me point to different but somewhat related problems: selfsimilar profiles for the Broadwell model with a relaxation time of the form $1/\varepsilon\, t$ in [56], existence of discrete traveling waves for relaxing schemes in [57–59], non-autonomous case, motivated by traffic modeling in the presence of a moving barrier which blocks a single lane, in [60].

3.2 Stability

A precise statement to quantify the strong form of attractivity of shock waves for (1.2) is provided in [61] where it is proved that, for convex fluxes, compactly supported perturbations of a shock waves are dissipated (by converting them into a space translation) in finite time. Stability persists for relaxation shocks and different approaches have been proposed to quantify it in a precise way.

A first standard step is to consider a frame moving with the speed σ of the relaxation shock, so that the traveling wave (ϕ, ψ) turns into a stationary solution. In addition, by considering an appropriate space translation of the relaxation shock, it can be assumed that

$$\int_{-\infty}^{+\infty} \big\{u(x,0) - \phi(x)\big\}\mathrm{d}x = 0. \tag{3.11}$$

In other words, without loss of generality, the initial datum can be assumed to be a zero mass perturbation. On the contrary, when the relaxation system has more than one conserved quantity, i.e. u is not scalar, such requirement is restrictive and prevents the formation of additional diffusion waves, with a consequent substantial simplification of the problem.

The conservative form of the first equation (1.1) guarantees that (3.11) persists for any positive time. This suggests to consider the decomposition $(u, v) = (\phi, \psi) + (\partial_x z, w)$, which gives raise to the perturbation system for the couple (z, w)

$$\begin{cases} \partial_t z + f(\phi + \partial_x z, \psi + w) - f(\phi, \psi) = 0, \\ \partial_t w + \partial_x \Big\{g(\phi + \partial_x z, \psi + w) - g(\phi, \psi)\Big\} = h(\phi + \partial_x z, \psi + w) - h(\phi, \psi). \end{cases} \tag{3.12}$$

The benefits of working with the integrated variable z, observed initially for conservation laws with viscosity (see the enlightening description in [62]), extends also to the relaxation case.

Going further, the coupling condition (1.6) guarantees that it is possible to determine, from the first equation (3.12), the component w as a function of the wave (ϕ, ψ) and z and its derivatives. Then, the second equation gives a scalar equation for z with the form

$$\partial_{tt} z + (\lambda_1 + \lambda_2)\partial_{tx} z + \lambda_1 \lambda_2 \partial_{xx} z - \partial_v h\big\{\partial_t z + \mu \partial_x z\big\} = \text{H.O.T.}, \tag{3.13}$$

where H.O.T. collects the super-linear terms and μ, called dynamic subcharacteristic speed, is

$$\mu = \partial_u f - \partial_v f \, \partial_u h \, \partial_v h^{-1}\Big|_{(\phi,\psi)} \tag{3.14}$$

Stability of weak relaxation shocks arise from dissipativity of the wave-type equation (3.13).

Theorem 3.3 (T.-P. Liu 1987 [1]) Assume the subcharacteristic condition (3.6). Then for shock size $|u_+ - u_-|$ and initial perturbation $(z_0, w_0) \in H^3 \times H^2$ sufficiently small, the solution (u, v) to (1.1) is global and converges to (ϕ, ψ) as $t \to +\infty$ in $L^\infty(\mathbb{R})$.

An effective tool to untangle the dissipation properties encoded in (3.13) is the energy method. A simplified and introductory presentation (in particular for the basic energy estimates in the constant coefficient case) can be found in [63] (pp.237–243). As noted in [1], for relaxation shocks, this technique works as consequence of three main features: relaxation structure, subcharacteristic regime, compressibility of the shock. The first two points are settled in assumptions (1.3) and (3.6) and, in principle, are not directly related to the small amplitude assumption. Compressibility refers to the sign of the space derivative of the dynamic subcharacteristic speed μ. For weak shocks, μ is close to its corresponding equilibrium value λ_*, given in (1.4), and appropriate conditions on f_* (such as convexity) allow to determine special sign properties in the energy estimates, as for conservation laws with viscosity.

Energy approach requires long computations and a delicate handling of the integral terms into play; as such, it is not particularly appealing. But, it is a very strong and flexible technique. In the case under study, by the same approach used to prove Theorem 3.3, P. Zingano [64] refined the stability result providing algebraic decay rates for initial perturbations with appropriate space-decay at infinity (see Theorem 3.1 in [64]). As in the case of conservation laws with viscosity, the method relies on the use of weighted energy inequalities.

For specific 2×2 systems, the energy method can be pushed forward and provide more general statements. The more explored case is the Jin–Xin model (1.8), for which nonlinear stability of large relaxation shocks has been proved in [32, 33], also in the case of nonconvex fluxes f_* with either $\sigma = f_*'(u_-)$ or $\sigma = f_*'(u_+)$, and in [65, 66], where, in addition, exponential decay of exponentially localized perturbations is shown.

Still for the Jin–Xin model, a dynamical systems approach, using in a fundamental way the special properties of the system as described in [6], is at the base of the L^1-stability result in [67]. By this method, it can be shown that large perturbations are dissipated by the system, but no decay rate is provided (and cannot be, in a purely L^1 framework). An attempt of collecting the energy method and the dynamical system approach has been performed in [68]. A different direction has been pursued in the recent contribution [69]. Here, based on abstract C_0-semigroup theory and spectral analysis, a nonlinear exponential stability result in the space $H^1_\alpha(\mathbb{R}) := \{(u, v) \,:\, (u, v) \cdot w_\alpha \in H^1(\mathbb{R})\}$ where $w_\alpha(x) = \cosh(\alpha x)$ is proved. Being based on a linearization principle, the analysis concerns with small perturbations; by the same token, it can be expected that the same strategy can be applied to more general 2×2 systems.

The energy approach can be adapted also to the p-system with relaxation (2.2). The small-amplitude case has been considered in [20], allowing the initial datum to be out-of-equilibrium as $x \to \pm\infty$ (as for rarefaction waves, this requires an additional correction terms for the space-asymptotic dynamics) and in [34], which deals also with the degenerate cases $f_*'(u_\pm) = \sigma$. A further promising step has been achieved in [69], where it is shown that the linearized operator at the relaxation shock generates an exponentially decaying semigroup in the framework of exponentially weighted space.

For larger relaxation system whose equilibrium model is a system the main crucial difference resides in the fact that the zero mass requirement (3.11) is a genuine restriction. Carrying non-zero mass perturbations generate both a shift in the reference relaxation shock and the appearance of additional diffusion waves, whose decay is crucial to prove nonlinear stability. Hence, using integrated variables is an effective restriction and provides a partial asymptotic stability result (see [35] for the Broadwell system, [39] for a reactive flow model, [70] for vectorial Jin–Xin system). The first result for general perturbations is probably [36] which deals with the Broadwell model. Following [71] for the case of conservation laws with viscosity, the solution $U = (u, v, w)$ to (3.8) is decomposed as

$$U(x,t) = \Phi(x) + U_1(x,t) + U_2(x,t) + U_*(x,t),$$

where Φ is the relaxation shock, U_1 and U_2 are, respectively a diffusive and a linear hyperbolic wave, U_* is the remainder. The construction is based on the Chapman–Enskog and thus requires a congenital small-amplitude assumption.

Theorem 3.4 (R.E. Caflisch and T.-P. Liu [36]) If the shock size $|U_+ - U_-|$ and the initial perturbation $|U_0 - \Phi|_{L^1 \cap H^2}$ are sufficiently small, then the solution to (3.8) is global and there exists x_0 such that U converges to $\Phi(\cdot + x_0)$ as $t \to +\infty$ in $L^\infty(\mathbb{R})$.

The problem of dealing with general perturbations has been considered also in [72] for the viscoelasticity model (3.9) and limited to the linear stability problem, meaning that the non-linear term $w_*(u)$ is substituted by $w_*(\phi) + w_*'(\phi)(u - \phi)$ with ϕ first component of the relaxation shock. The strategy is to apply energy estimate after having decomposed the solution $U = (u, v, w)$ to (3.9) as a superposition of the form

$$U(x,t) = \Phi(x) + \frac{A}{\sqrt{t+1}}\, m\left(\frac{x - \lambda(t+1)}{\sqrt{t+1}}\right) + \frac{A_1}{t+1}\, m_1\left(\frac{x - \lambda(t+1)}{\sqrt{t+1}}\right) + U_*(x,t),$$

where Φ is the shock wave, m a diffusion wave, m_1 a high-order correction, A, A_1, λ appropriate coefficients and U_* the remainder. The vector A is chosen so that the second term in the decomposition carries the excess mass caused by the initial perturbation. The same strategy has been applied in [73] to obtain an analogous linear stability result for the reactive flow model considered in [39]. All the results listed in this and in the previous paragraph, except for [35], are limited to the case of small-amplitude relaxation shocks.

The use of diffusive correctors furnishes a detailed description of the asymptotic behavior, but it is apparently too intricated to permit the treatment of general relaxation systems. A different approach consists in studying the linearized operator $\mathcal{L}$ at the profile Φ, obtaining precise estimates on the semigroup $e^{\mathcal{L}t}$, and deduce nonlinear stability by means of Duhamel formula analysis and energy estimates. Since the linearized operator $\mathcal{L}$ have variable coefficients, it is natural to apply Laplace transform and reduce to the corresponding resolvent system. Such approach has been exploited in [46, 74] and will be shortly reviewed in the following lines (see [75] for a slightly extended presentation). For brevity, general structural assumptions on the system and on the relaxation profile will be assumed without entering the details.

For general relaxation systems, the operator $\mathcal{L}$ enjoys some significant properties: $\lambda = 0$ belongs to the spectrum of $\mathcal{L}$, under reasonable assumptions (on the equilibrium system and on the heteroclinic connection for Φ) the geometric multiplicity of 0 is one, the essential spectrum of $\mathcal{L}$ is "well-behaved" for stable asymptotic states $U_{\pm}$. These suggest a fascinating connection between nonlinear stability and the spectral stability assumption

$$\sigma_{\rm p}(\mathcal{L}) \cap \{{\rm Re}\lambda \geq 0\} = \{0\}, \tag{3.15}$$

where $\sigma_{\rm p}$ denotes the pointwise spectrum.

In [46], it is proved that the Green distribution $\mathcal{G}$ for the linearized operator $\mathcal{L}$ can be decomposed into the sum of three terms, $\mathcal{G} = \mathcal{E} + \tilde{\mathcal{G}} + \mathcal{H}$, where $\mathcal{E}$ contains informations relative to the shift of the wave, the term $\tilde{\mathcal{G}}$ is approximately a sum of gaussian signals scattered by the shock layer and the term $\mathcal{H}$ contains delta distributions –exponentially decaying in time– describing propagation of signals along hyperbolic characteristics. This decomposition shows that the relaxation shocks are linearly stable whenever the spectral stability condition (3.15) (and the additional structural assumptions) are satisfied. Since the large time behavior is mainly determined by the term $\tilde{\mathcal{G}}$, rates of decay are of heat kernel type.

Combining the additional stratagem to introduce a shift variable $\delta = \delta(t)$ in the perturbation equation to get rid of the non-decaying term $\mathcal{E}$ and an appropriate nonlinear iteration argument –compounding linearized decay rates and energy estimates– permits to prove the following general nonlinear stability result.

Theorem 3.5 (C. Mascia and K. Zumbrun [74]) Under structural assumptions and the spectral stability condition (3.15), if the initial perturbation $W_0 := U_0 - \Phi$ is sufficiently small in $L^1 \cap H^2$, the solution U is global and satisfies

$$|U(\cdot,t) - \Phi(\cdot - \delta(t))|_{L^p} \leq C|U_0 - \Phi|_{L^1\cap H^2}(1+t)^{-\frac{1}{2}\left(1-\frac{1}{p}\right)} \qquad p \in [1,+\infty]$$

for some bounded function δ with derivative bounded by $C|W_0|_{L^1\cap H^2}(1+t)^{-1/2}$.

The optimal decay rates are obtained by the detailed description of the Green function itself, considered as the inverse Laplace transform of the kernel of the integral operator defined by the resolvent equation. Indeed, renouncing to the detailed description provided by diffusion waves have the notable advantage of providing a statement valid in a very general framework.

Nonlinear stability is now reduced to the verification of the spectral stability property (3.15). A series of papers contributed to enlarge the class of systems for which such assumption holds: [76] deals with (vectorial) Jin–Xin systems, [77] with relaxation models with a non-degenerate jacobian of the flux, [78] with general relaxation models. All of these results applies to the case of small amplitude relaxation shocks.

The extension to the large amplitude case is not obvious and strongly system dependent. Indeed, based on the necessary conditions for linear stability exploited in [79] by means of the Evans function approach, it has been shown in [80] the existence of unstable relaxation shocks. The construction is explicit and concerns with a specific 4×4 Jin–Xin systems with a two-dimensional equilibrium system with flux described by fourth-order polynomials. As for positive results, since stability with respect to zero–mass perturbations implies the spectral stability condition, Theorem 3.5 applies to large amplitude shocks for the Broadwell system, thanks to the result in [35]. The literature does not provide other examples of stable or unstable large amplitude relaxation shocks and the understanding on the frontier between stability and instability is still very unclear at the moment.

Coming back to the comprehensive description, a relevant direction is the stability of multi-dimensional relaxation shocks, explored since [81], which analyzed the case of the Jin–Xin system in two dimensions. The same system has been considered in [82], while the classical Saint Venant's shallow river model is the case under study in [63]. General relaxation models has been tackled in [83] (scalar equilibrium equation) and in [84] (vectorial equilibrium equation).

To complete the outlook on the stability problem, it is worthy to quote [58, 59, 85] on the stability of discrete relaxation shocks, [54, 55] relative to the case of compresence of diffusion and relaxation, [86] dealing with the stability of contact discontinuities for the Jin–Xin model, and [87] concerning a specific model where stability of large discontinuous relaxation shocks can be proved. The last paper is one of the few concerning stability in presence of a jump, but it should be stressed that model is very specific since it can be reduced to a triangular system and thus, essentially, to a problem in scalar conservation laws (with source).

4 Wave Patterns and Subshocks

The last Section of [1] is dedicated to the analysis of wave patterns, that are combinations of nonlinear waves (rarefaction and shocks) which tend to an exact solution of (1.2) and becomes noninteracting as $t \to +\infty$. Also in this context, subcharacteristic condition plays a crucial role. To start with, let us assume that (1.5) is valid for all the values (u, v) under consideration.

Following the discussion of the previous Sections:

— a relaxation rarefaction of (1.1) is a couple (r, s) where r is a rarefaction wave of (1.2) corresponding to a Riemann problem determined by (2.4) and $s := v_*(r)$;

— a relaxation shock of (1.1) is a traveling wave solution (ϕ, ψ) with asymptotic states $(u_\pm, v_*(u_\pm))$.

The two classes of waves are considered as connections of the state $(u_-, v_*(u_-))$ with $(u_+, v_*(u_+))$. Note that, even if a rarefaction wave is not an exact solution of the relaxation system, the error decays to zero as $t \to +\infty$ and thus it is an asymptotic exact solution. A rarefaction has a span of propagation speeds bounded by the extremal speeds $\lambda_*(u_\pm)$ while a relaxation shock has a definite speed σ given by the Rankine–Hugoniot condition.

A (finite) non-interacting wave pattern is a set $\{u_1 < u_2 < \cdots < u_N\}$ such that

— to each interval $[u_k, u_{k+1}]$ it is associated either a rarefaction wave or a shock;

— the corresponding (rarefaction/shock) speeds are ordered.

A possible statement for the analysis proposed in [1] is the following.

Theorem 4.1 Given $u_- < u_+$, let the flux f_* have a finite number of changes of convexity in $[u_-, u_+]$ and let $f_*^{\rm conv}$ be the convex envelope of f_* in $[u_-, u_+]$. If (1.5) holds for any $u \in [u_-, u_+]$ and v such that $f(u, v) = f_*^{\rm conv}(u)$ then f_* defines a noninteracting wave pattern by linking rarefactions to intervals where $f_*^{\rm conv} = f_*$ and shocks to intervals where $f_*^{\rm conv} < f_*$.

To substantiate the assertion, consider the new variable $w = f(u, v)$ so that the manifold $\mathcal{E}_*$ in the plane (u, v) is transformed in the graph of function f_* in the plane (u, w). Then, recalling the discussion of the previous Sections, it is readily seen that each interval where $f_*^{\rm conv} = f_*$ corresponds to a well-defined rarefaction wave for (1.2), and hence for (1.1), and each interval where $f_*^{\rm conv} < f_*$ defines a relaxation shock, as previously described.

The analysis of nonlinear stability of non-interacting wave pattern suffers of many problems in its formulation itself –starting from the fact that a pattern is not an exact solution– and has not yet been addressed. Related results has been proved in the case of the 2×2 Jin–Xin system (1.8) for an initial-boundary value problem in the quarter plane. Specifically, [88] discusses nonlinear stability of rarefaction waves, steady states and non-interacting combination of the two kind of solutions; [89] is devoted to the case of a superposition of two traveling waves and the stability is complemented with decay rate analysis. A different problem is considered in [90], which concerns with the case of interaction of shocks for the Jin–Xin system.

A further step forward requires the analysis of relaxation shocks outside the subcharacteristic regime. This is discussed in [1] with no many details, so that the reading is laborious and even establishing a precise statement is uneasy. In general, presence of non-subcharacteristic regions generates the presence of a discontinuity in such a way that the supercharacteristic regime is istantaneously by-passed.

A sub-shock of a relaxation shocks is a discontinuity in the wave profile which defines an admissible jump for the principal part of system (1.1), namely

$$\partial_t U + \partial_x F(U) = 0 \qquad \text{with} \quad U = (u, v), \quad F = (f, g).$$

To fix the notation, given a state $U_0 = (u_0, v_0)$, let $H(U_0)$ be the Hugoniot locus

$$H(U_0) := \{U = (u, v) \,:\, \sigma\,(U - U_0) = F(U) - F(U_0) \ \text{ for some } \sigma \in \mathbb{R}\}\,. \tag{4.1}$$

Given $U \in H(U_0)$, let $\sigma(U_0; U)$ be the corresponding value of σ in the definition of (4.1).

In a neighborhood of U_0, the set $H(U_0)$ is the union of two curves, $H_i(U_0)$ with $i = 1, 2$ corresponding to values of σ close to the eigenvalue λ_i. Under the assumption that the two curves $H_i(U_0)$ are globally connected, let $H_i(U_\ell, U_r)$ be the (connected) arc in $H_i(U_\ell)$ bounded by U_r. The admissibility condition for a jump from U_ℓ to U_r is

$$\sigma(U_\ell; U_r) \le \sigma(U_\ell; U) \qquad \text{for any} \quad U \in H_i(U_\ell, U_r). \tag{4.2}$$

With this terminology at hand, it is feasable to review the construction in [1] for possibly discontinuous relaxation shocks.

Assume attractivity (1.3), coupling (1.6) for any (u, v) under consideration. Let u_-, u_+ be such that the corresponding shock solution (3.1) is admissible for the conservation law (1.2) and, without loss of generality, suppose $u_- < u_+$, $\sigma = f\big(u_-, v_*(u_-)\big) = 0$ and that the subcharacteristic condition (1.5) holds on $\mathcal{E}_*$ for any $u \in [u_-, u_+]$. In addition, denoted by Ψ the function defined by $f(u, v) = 0$ and such that $\Psi(u_-) = v_-$, assume

$$\partial_v f > 0 \qquad \text{and} \qquad \gamma(u) := g(u, \Psi(u)) > g\big(u_-, v_*(u_-)\big) \quad \forall\, u \in (u_-, u_+).$$

Also, let the extrema of γ be non degenerate, i.e. if $\gamma'(u) = 0$, then $\gamma''(u) \ne 0$. Finally, assume that the subcharacteristic condition (1.5) can be violated only by the trespassing of 0 by λ_2, that is $\lambda_1 \le -c_0 < 0$ for any (u, v). Then it is possible to shape a relaxation shock with asymptotic states $\big(u_\pm, v_*(u_\pm)\big)$ with eventual jumps that are admissible at least if they are small.

As shown in Section 3, when it is smooth, the component ϕ satisfies

$$\lambda_1\, \lambda_2\Big|_{(\phi,\Psi(\phi))} \frac{\mathrm{d}\phi}{\mathrm{d}\xi} = -h\, \partial_v f\Big|_{(\phi,\Psi)},$$

where $\Psi = \Psi(v)$ is implicitly defined by the equality $f(u, \Psi(u)) = 0$.

Admissibility of the pure shock u_-, u_+ is equivalent to the requirement $h\,\partial_v f(\phi, \Psi(\phi)) > 0$ for any $\phi \in (u_-, u_+)$. Then, for $\gamma(u) := g\big(u, \Psi(u)\big)$, there holds

$$\frac{\mathrm{d}\gamma}{\mathrm{d}u} = \left(\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)^{-1} \frac{\mathrm{d}\gamma}{\mathrm{d}x} = -\frac{\lambda_1\,\lambda_2}{\partial_v f}\Big|_{(\phi,\Psi)}.$$

Hence, the function γ is increasing when $\lambda_1\lambda_2 < 0$ and decreasing otherwise. In particular, subcharacteristic assumptions at the equilibria $(u_\pm, v_*(u_\pm))$ implies that the function γ is increasing in a neighborhood of the endstates u_- and u_+.

Then, using the auxiliary function γ_0, upper bound of all the monotone increasing functions bounded from above by γ and defined by

$$\gamma_0(u) := \min_{s\in[u,u_+]} \gamma(s) \qquad u \in [u_-, u_+],$$

the profile of the relaxation shock can be defined as follows: where γ_0 is strictly increasing, ϕ is smooth and satisfies equation (3.7), otherwise the solution exhibit a jump.

It remains to show that the jumps are shocks for the system (1.1). Let (u_ℓ, v_ℓ) and (u_r, v_r) be the left/right states of the jump. Rankine–Hugoniot conditions are equivalent to continuity of both f and g along the jump, which is guaranteed by the membership to the graph of Φ of both $(u_{\ell,r}, v_{\ell,r})$ and the constancy of the function γ_0 in $[u_\ell, u_r]$, respectively.

To check the admissibility condition, it has to be shown that for any couple (u, v) in the Hugoniot locus of (u_ℓ, v_ℓ) and with $u_\ell < u < u_r$, there holds $\sigma(u_\ell, v_\ell; u, v) \geq 0$. By the non-degeneracy assumption, $\gamma'(u_\ell) > 0$. Hence, $\lambda_1(u_\ell, v_\ell) < 0 < \lambda_2(u_\ell, v_\ell)$. For small jumps, $\sigma(u_\ell, v_\ell; u, v) \approx \lambda_1(u_\ell, v_\ell)$ along $H_1(u_\ell, v_\ell)$. Since $\lambda_1\lambda_2(u_r, v_r) = 0$ and $\lambda_1 \leq -c_0 < 0$, we infer that $(u_r, v_r) \in H_2(u_\ell, v_\ell)$.

Next, let us travel along $H_2(u_\ell, v_\ell; u_r, v_r)$ toward (u_r, v_r). For $u > u_\ell$ and close to u_ℓ, the value $\sigma(u_\ell, v_\ell; u, v)$ is close to $\lambda_2(u_\ell, v_\ell) > 0$. Let (u_*, v_*) the first point in $H_2(u_\ell, v_\ell; u_r, v_r)$ such that $\sigma(u_\ell, v_\ell; u, v) = 0$. By the Rankine–Hugoniot conditions, $f(u_*, v_*) = 0$ and $g(u_*, v_*) = g(u_\ell, v_\ell)$. Hence, $v_* = \Psi(u_*)$ and, by definition of γ, $u_* = u_r$. Thus, σ is positive in between u_ℓ and u_r and the admissibility is proved. Let me remark the only point where the smallness assumption of the jump is needed is to show that (u_r, v_r) belongs to H_2.

A general analysis of existence and internal structure of large relaxation shocks has not yet been accomplished. Specific cases of relaxation shocks with jumps are rare if not completely missing. The stability analysis in such a case is still at the starting point since both energy approach and Green function estimates use in a crucial way the smoothness of the wave profile.

5 A Sort of Conclusion

This bird's-eye view on the evolution of the topics discussed in [1] suggests a main (very wide) issue to be explored in the analysis of conservation laws with relaxation: the comprehension of the dynamics generated by the relaxation mechanisms for <u>large</u> solutions. With the single exception of the 2×2 Jin–Xin model, almost all the available results concerns with small initial data. Complete generality is out-of-reach and it would be already instructive to follow the approach employed in [1] and restrict the attention to general "small" systems and analyze the possible behaviors for solutions large and/or far from equilibrium regimes, with specific attention to critical phenomena such as disappearance of traveling waves or loss of stability.

Stability of large rarefaction waves in the general quasi-linear case is not much explored and almost nothing is known in a weak framework (when standard energy estimates are not applicable). A wider understanding on existence of large relaxation shocks in presence of subshocks would be relevant as well as the development of some flexible technique to analyze stability of discontinuous traveling waves. Comparing viscous and relaxation dissipation –which are known to be more or less equivalent in the small-amplitude regime– in the case of large solutions is a fascinating direction, which could elucidate when the approach of extending the number of unknowns to describe the far-from-equilibrium dynamics generates totally different results with respect to other higher order regularizations. This last point could contribute to the discussion on the adequacy of hyperbolic models as correctors of simplified parabolic ones, in the spirit of the Extended Thermodynamics, and in the possibility for such extended models to exhibit new phenomena not predicted by other higher-order regularizations (see [91, 92]).

Concluding, there is still much that can be explored after the 1987 Tai-Ping Liu's paper and the number of citations of [1] is expected to increase further in the future years confirming the foundational role of "Hyperbolic conservation laws with relaxation" in the development of theory of hyperbolic conservation laws.

Acknowledgements I am thankful to D. Aregba-Driollet, B. Hanouzet, T. Luo, P. Marcati, R. Natalini, W.-A. Yong, K. Zumbrun for some (electronic) interviews on specific topics discussed in the paper. Even if, for the sake of shortness, I had to select drastically the hints I received, they greatly helped me in finding a coherent orientation in this mare magnum of information.

References

[1] Liu T-P. Hyperbolic conservation laws with relaxation. Comm Math Phys, 1987, **108**(1): 153–175

[2] Whitham J. Linear and Nonlinear Waves. New York: Wiley, 1974

[3] Chen G Q, Liu T-P. Zero relaxation and dissipation limits for hyperbolic conservation laws. Comm Pure Appl Math, 1993, **46**(5): 755–781

[4] Chen G Q, Levermore C D, Liu T-P. Hyperbolic conservation laws with stiff relaxation terms and entropy. Comm Pure Appl Math, 1994, **47**(6): 787–830

[5] Jin S, Xin Z. The relaxation schemes for systems of conservation laws in arbitrary space dimensions. Comm Pure Appl Math, 1995, **48**(3): 235–277

[6] Natalini R. Convergence to equilibrium for the relaxation approximations of conservation laws. Comm Pure Appl Math, 1996, **49**(8): 795–823

[7] Chern I-L. Long-time effect of relaxation for hyperbolic conservation laws. Commun Math Phys, 1995, **172**: 39–55

[8] Liu H, Natalini R. Long-time diffusive behavior of solutions to a hyperbolic relaxation system. Asymptot Anal, 2001, **25**(1): 21–38

[9] Ueda Y, Kawashima S. Large time behavior of solutions to a semilinear hyperbolic system with relaxation. J Hyperbolic Differ Equ, 2007, **4**(1): 147–179

[10] Yao Z-A, Zhu C. L^p-convergence rate to diffusion waves for p-system with relaxation. J Math Anal Appl, 2002, **276**(2): 497–515

[11] Ruggeri T, Serre D. Stability of constant equilibrium state for dissipative balance laws system with a convex entropy. Quart Appl Math, 2004, **62**(1): 163–179

[12] Bianchini S, Hanouzet B, Natalini R. Asymptotic behavior of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. Comm Pure Appl Math, 2007, **60**(11): 1559–1622

[13] Deng S, Wang W. Pointwise estimates of solutions for the multi-dimensional scalar conservation laws with relaxation. Discrete Contin Dyn Syst, 2011, **30**(4): 1107–1138

[14] Escobedo M, Laurençot Ph. Asymptotic behaviour for a partially diffusive relaxation system. Quart Appl Math, 2003, **61**(3): 495–512

[15] Zhao H. Nonlinear stability of strong planar rarefaction waves for the relaxation approximation of conservation laws in several space dimensions. J Differential Equations, 2000, **163**(1): 198–222

[16] Matsumura A, Nishihara K. Global stability of the rarefaction waves of a one-dimensional model system for compressible viscous gas. Comm Math Phys, 1992, **144**(1992): 325–335

[17] Liu H. The L^p stability of relaxation rarefaction profiles. J Differential Equations, 2001, **171**(2): 397–411

[18] Wu J, Zheng Y. Some results on hyperbolic systems with relaxation. Acta Mathematica Scientia, 2006, **26**(4): 767–780

[19] Zhang Y, Tan Z, Sun M-B. Zero relaxation limit to centered rarefaction waves for Jin-Xin relaxation system. Nonlinear Anal, 2011, **74**(6): 2249–2261

[20] Zhu C. Asymptotic behavior of solutions for p-system with relaxation. J Differential Equations, 2002, **180**(2): 273–306

[21] Zhao H, Zhao Y. Convergence to strong nonlinear rarefaction waves for global smooth solutions of p-system with relaxation. Discrete Contin Dyn Syst, 2003, **9**(5): 1243–1262

[22] Yang T, Zhu C. Existence and non-existence of global smooth solutions for p-system with relaxation. J Differential Equations, 2000, **161**(2): 321–336

[23] Luo T. Asymptotic stability of planar rarefaction waves for the relaxation approximation of conservation laws in several dimensions. J Differential Equations, 1997, **133**(2): 255–279

[24] Zou Q, Zhao H, Wang T. The Jin-Xin relaxation approximation of scalar conservation laws in several dimensions with large initial perturbation. J Differential Equations, 2012, **253**(2): 563–603

[25] Hsiao L, Pan R. Nonlinear stability of rarefaction waves for a rate-type viscoelastic system. Chinese Ann Math Ser B, 1999, **20**(2): 223–232

[26] Hsiao L, Pan R. Zero relaxation limit to centered rarefaction waves for a rate-type viscoelastic system. J Differential Equations, 1999, **157**(1): 20–40

[27] Wang W C. Asymptotic towards rarefaction wave of the Jin-Xin relaxation model for the p-system. IMS Conference on Differential Equations from Mechanics (Hong Kong, 1999). Methods Appl Anal, 2001, **8**(4): 689–701

[28] Wang W C. Nonlinear stability of centered rarefaction waves of the Jin-Xin relaxation model for 2×2 conservation laws. Electron J Differential Equations, 2002, **57**: 20 pp

[29] Nishihara K, Zhao H, Zhao Y. Global stability of strong rarefaction waves of the Jin-Xin relaxation model for the p-system. Comm Partial Differential Equations, 2004, **29**(9/10): 1607–1634

[30] Fan H, Luo T. Convergence to equilibrium rarefaction waves for discontinuous solutions of shallow water wave equations with relaxation. Quart Appl Math, 2005, **63**(3): 575–600

[31] Lambert W, Marchesin D. Asymptotic rarefaction waves for balance laws with stiff sources. Acta Mathematica Scientia, 2009, **29B**(6): 1613–1628

[32] Liu H, Woo C W, Yang T. Decay rate for travelling waves of a relaxation model. J Differential Equations, 1997, **134**(2): 343–367

[33] Liu H, Wang J, Yang T. Stability of a relaxation model with a nonconvex flux. SIAM J Math Anal, 1998, **29**(1): 18–29

[34] Li T, Liu H. Stability of a traffic flow model with nonconvex relaxation. Commun Math Sci, 2005, **3**(2): 101–118

[35] Kawashima S, Matsumura A. Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion. Comm Math Phys, 1985, **101**(1): 97–127

[36] Caflisch R E, Liu T-P. Stability of shock waves for the Broadwell equations. Commun Math Phys, 1988, **114**: 103–130

[37] Tzavaras A E. Materials with internal variables and relaxation to conservation laws. Arch Ration Mech Anal, 1999, **146**(2): 129–155

[38] Wu Y, Liu I-S. Shock structure in viscoelasticity of relaxation type. Nonlinear Anal, 2006, **65**(4): 785–794

[39] Pan R. The nonlinear stability of travelling wave solutions for a reacting flow model with source term. Acta Mathematica Scientia, 1999, **19**(1): 26–36

[40] Chalons C, Coulombel J-F. Relaxation approximation of the Euler equations. J Math Anal Appl, 2008, **348**: 872–893

[41] Xie W. Shock profile for gas dynamics in thermal nonequilibrium. Involve, 2013, **6**(3): 311–322

[42] Aregba-Driollet D, Hanouzet B. Kerr-Debye relaxation shock profiles for Kerr equations. Commun Math Sci, 2011, **9**(1): 1–31

[43] Torrilhon M. Characteristic waves and dissipation in the 13-moment-case. Continuum Mech Thermodynamics, 2000, **12**(5): 289–301

[44] Simić S. Shock structure in continuum models of gas dynamics: stability and bifurcation analysis. Nonlinearity, 2009, **22**(6): 1337–1366

[45] Yong W-A, Zumbrun K. Existence of relaxation shock profiles for hyperbolic conservation laws. SIAM J Appl Math, 2000, **60**(5): 1565–1575

[46] Mascia C, Zumbrun K. Pointwise Green's function bounds and stability of relaxation shocks. Indiana Univ Math J, 2002, **51**(4): 773–904

[47] Dressel A, Yong W-A. Existence of traveling-wave solutions for hyperbolic systems of balance laws. Arch Ration Mech Anal, 2006, **182**(1): 49–75

[48] Métivier G, Zumbrun K. Existence of semilinear relaxation shocks. J Math Pures Appl, 2009, **92**: 209–231

[49] Métivier G, Texier B, Zumbrun K. Existence of quasilinear relaxation shock profiles in systems with characteristic velocities. Ann Fac Sci Toulouse Math, 2012, **21**(1): 1–23

[50] Métivier G, Zumbrun K. Existence and sharp localization in velocity of small-amplitude Boltzmann shocks. Kinet Relat Models, 2009, **2**(4): 667–705

[51] Liu T-P, Yu S-H. Boltzmann equation: micro-macro decompositions and positivity of shock profiles. Commun Math Phys, 2004, **246**: 133–179

[52] Bernhoff N, Bobylev A. Weak shock waves for the general discrete velocity model of the Boltzmann equation. Commun Math Sci, 2007, **5**(4): 815–832

[53] Jin S, Liu J G. Relaxation and diffusion-enhanced dispersive waves. Proc R Soc Lond A, 1994, **446**(1928): 555–563

[54] Li T. Stability of traveling waves in quasi-linear hyperbolic systems with relaxation and diffusion. SIAM J Math Anal, 2008, **40**(3): 1058–1075

[55] Wang L, Wu Y, Li T. Exponential stability of large-amplitude traveling fronts for quasi-linear relaxation systems with diffusion. Phys D, 2011, **240**(11): 971–983

[56] Slemrod M, Tzavaras A E. Shock profiles and self-similar fluid dynamic limits. Proceedings of the Second International Workshop on Nonlinear Kinetic Theories and Mathematical Aspects of Hyperbolic Systems (Sanremo, 1994). Transport Theory Statist Phys, 1996, **25**(3/5): 531–541

[57] Liu H, Wang J, Yang T. Existence of the discrete travelling waves for a relaxing scheme. Appl Math Lett, 1997, **10**(3): 117–122

[58] Liu H, Wang J, Yang T. Nonlinear stability and existence of stationary discrete travelling waves for the relaxing schemes. Japan J Indust Appl Math, 1999, **16**(2): 195–224

[59] Ye M. Existence and asymptotic stability of relaxation discrete shock profiles. Math Comp, 2004, **73**(247): 1261–1296

[60] Chou S-W, Hong J M, Lin Y-C. Existence and large time stability of traveling wave solutions to nonlinear balance laws in traffic flows. Commun Math Sci, 2013 **11**(4): 1011–1037

[61] Liu T-P. Invariants and asymptotic behavior of solutions of a conservation law. Proc Amer Math Soc, 1978, **71**(2): 227–231

[62] Goodman J. Nonlinear asymptotic stability of viscous shock profiles for conservation laws. Arch Ration Mech Anal, 1986, **95**(4): 325–344

[63] Ha S-Y, Yu S-H. Wave front tracing and asymptotic stability of planar travelling waves for a two-dimensional shallow river model. J Differential Equations, 2002, **186**(1): 230–258

[64] Zingano P R. Nonlinear stability with decay rate for traveling wave solutions of a hyperbolic system with relaxation. J Differential Equations, 1996, **130**(1): 36–58

[65] Mei M, Yang T. Convergence rates to travelling waves for a nonconvex relaxation model. Proc Roy Soc Edinburgh Sect A, 1998, **128**(5): 1053–1068

[66] Ueda Y. Stability of travelling wave solutions to a semilinear hyperbolic system with relaxation. Math Methods Appl Sci, 2009, **32**(4): 419–434

[67] Mascia C, Natalini R. L^1 nonlinear stability of traveling waves for a hyperbolic system with relaxation. J Differential Equations, 1996, **132**(2): 275–292

[68] Zhang H, Zhao Y. Convergence rates to travelling waves for a relaxation model with large initial disturbance. Acta Mathematica Scientia, 2004, **24**(2):213–227.

[69] Li T, Wu Y. Linear and nonlinear exponential stability of traveling waves for hyperbolic systems with relaxation. Commun Math Sci, 2009, **7**(3): 571–593

[70] Liu H. Asymptotic stability of relaxation shock profiles for hyperbolic conservation laws. J Differential Equations, 2003, **192**(2): 285–307

[71] Liu T-P. Nonlinear stability of shock waves for viscous conservation laws. Mem Am Math Soc, 1985, **56**(328): 1–109

[72] Luo T, Serre D. Linear stability of shock profiles for a rate-type viscoelastic system with relaxation. Quart Appl Math, 1998, **56**(3): 569–586

[73] Hsiao L, Pan R. The linear stability of traveling wave solutions for a reacting flow model with source term. Quart Appl Math, 2000, **58**(2): 219–238

[74] Mascia C, Zumbrun K. Stability of large-amplitude shock profiles of general relaxation systems. SIAM J Math Anal, 2005, **37**(3): 889–913

[75] Mascia C. Stability and instability issues for relaxation shock profiles//Benzoni-Gavage S, Serre D, eds. Hyperbolic Problems: Theory, Numerics, Applications. Springer, 2008: 173–185

[76] Humpherys J. Stability of Jin-Xin relaxation shocks. Quart Appl Math, 2003, **61**(2): 251–263

[77] Plaza R, Zumbrun K. An Evans function approach to spectral stability of small-amplitude shock profiles, Discrete Contin Dyn Syst, 2004, **10**(4): 885–924

[78] Mascia C, Zumbrun K. Spectral stability of weak relaxation shock profiles. Comm Partial Differential Equations, 2009, **34**(2): 119–136

[79] Godillon P. Linear stability of shock profiles for systems of conservation laws with semi-linear relaxation. Phys D, 2001, **148**(3/4): 289–316

[80] Godillon P, Lorin E. A Lax shock profile satisfying a sufficient condition of spectral instability. J Math Anal Appl, 2003, **283**(1): 12–24

[81] Luo T, Xin Z. Nonlinear stability of shock fronts for a relaxation system in several space dimensions. J Differential Equations, 1997, **139**(2): 365–408

[82] Liu H. Asymptotic decay to relaxation shock fronts in two dimensions. Proc Roy Soc Edinburgh Sect A, 2001, **131**(6): 1385–1410

[83] Kwon B, Zumbrun K. Asymptotic behavior of multidimensional scalar relaxation shocks. J Hyperbolic Differ Equ, 2009, **6**(4): 663–708

[84] Kwon B. Stability of planar shock fronts for multidimensional systems of relaxation equations. J Differential Equations, 2011, **251**(8): 2226–2261

[85] Liu H. Convergence rates to the discrete travelling wave for relaxation schemes. Math Comp, 2000, **69**(230): 583–608

[86] Huang F, Pan R, Wang Y. Stability of contact discontinuity for Jin-Xin relaxation system. J Differential Equations, 2008, **244**(5): 1114–1140

[87] Li T. L^1 stability of conservation laws for a traffic flow model. Electron J Differential Equations, 2001: 1418

[88] Yang T, Zhao H, Zhu C. Asymptotic behavior of solutions to a hyperbolic system with relaxation and boundary effect. J Differential Equations, 2000, **163**(2): 348–380

[89] Hsiao L, Li H, Mei M. Convergence rates to superposition of two travelling waves of the solutions to a relaxation hyperbolic system with boundary effects. Math Models Methods Appl Sci, 2001, **11**(7): 1143–1168

[90] Lattanzio C, Serre D. Shock layers interactions for a relaxation approximation to conservation laws. NoDEA Nonlinear Differential Equations Appl, 1999, **6**(3): 319–340

[91] Geroch R. On hyperbolic "theories" of relativistic dissipative fluids. http://arxiv.org/abs/gr-qc/0103112v1

[92] Herrera L, Pavón D. Hyperbolic theories of dissipation: Why and when do we need them? Physica A, 2002, **307**: 121–130