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THEORY OF COLLABORATION AND COLLABORATIVE MEASURES

L. EGGHE
LUC, Universitaire Campus, B-3590 Diepenbeek, Belgium*
UIA, Universiteitsplein 1, B-2610 Wilrijk, Belgium

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Abstract—The paper discusses earlier attempts by Ajiferuke, Burrell, & Tague and by Englisch to define a single measure of collaboration. We show that the variables used in these papers are too rough and propose finer variables. We then formulate collaborative principles that good collaborative measures should satisfy and examine existing measures. Collaborative principles are designed in such a way that measures satisfying these principles can distinguish between (i.e., have different values for) different collaborative situations. We then present new collaborative measures better than the existing ones, in the sense that they satisfy all the studied collaborative principles. Many examples are presented and practical calculations are executed.

1. INTRODUCTION

Collaboration is usually expressed through a set of co-authored papers (e.g., written by a research team or another group such as a university, research center, etc.), but also relates in general to the cooperation or relations between individuals in social groups. In this paper we will not restrict ourselves to any viewpoint, but work within the general but clear frame of "boxes" in which one has "objects," as in Fig. 1. Here one has that (in the terminology of co-authored papers) there is one paper with three authors (labelled 1,2,3), three papers with two authors (respectively, 3,4; 1,3; and 5,6), and three papers with one author (namely 3, 4, and 6).

Intuitively, the situation as described in Fig. 1 looks more cooperative than the situation in Fig. 2, although each individual contributes to the same number of papers. How are we to quantify "degrees" of collaboration? How "fine" must our variables be in order to distinguish as much as possible among different collaborative situations?

In [1] (based on preliminary work of [6] and [7]), an attempt has been made to define a measure of collaboration as follows: Let f_j $(j = 1,2,\ldots,k)$ be the number of j-authored papers (in a certain discipline and a certain period of time) and let N be the total number of papers. Then the "collaborative coefficient" CC is defined as

$$CC = 1 - \frac{\sum_{j=1}^{q} \frac{1}{j} f_j}{N}, \tag{1}$$

where q denotes the maximal j such that $f_j \neq 0$. It was shown in [1] that this measure satisfies several "good" properties: $0 \leqq CC \leqq 1$, $CC = 0$ if there are only single-authored papers and it distinguishes between different levels j of multiple authorship. We remark here that the numbers $(f_j)_{j=1,\ldots,k}$ represent the dual of the more well-known distribution of Lotka: For Lotka, g_j = number of authors with j papers, whereas f_j deals with the number of papers with j authors [3],[4].

In [5], the collaborative problem is studied from a different point of view. Here the variables are $x_{i,i'}$, where $x_{i,i'}$ denotes the number of times author i has published a paper with author i' $(i,i' = 1,\ldots,n;\ i \neq i')$. Note that for every $i \neq i'$, $x_{i,i'} = x_{i',i}$. With these vari-

*Permanent address.

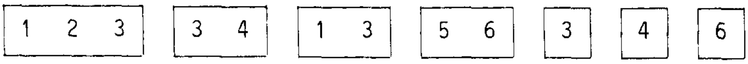

Fig. 1. Example of a collaborative system.

ables, several collaborative measures are considered. The simplest of these (but the same properties as all the others introduced in [5]) is

$$s = \left(\sum_{\substack{i,i'=1 \\ i \neq i'}}^{n} \sqrt{x_{i,i'}} \right)^2 \tag{2}$$

where n denotes the total number of different authors. Englisch has two main requirements for collaborative measures: the function must be concave and increasing in $x_{i,i'}$. The latter property will not be retained because this is more an aspect of productivity.

Why concavity is required will be explained further on, but with regard to another function, since we will show that both function (2) and function (1) are too limited to distinguish between different collaborative situations. In the next section we will introduce finer variables that can be used.

In the third section we formulate eight collaboration principles that collaborative measures must satisfy in order to be good measures.

Then, in the fourth section, the two measures CC and s as well as two new measures f and h are studied. Normalization is needed to have values less than one. These normalized measures are denoted by CC^*, s^*, f^*, and h^*. CC^* and f^* each fail on two important principles, while s^* and h^* fail on only one principle. However, based on s^* and h^*, two new measures γ_1 and γ_2 are constructed that satisfy *all* eight principles. It is furthermore shown that these measures give a very good balance of scores between 0 and 1, around 0.5. We will also give many explicit calculations of these measures.

Section five then summarizes the most important results.

II. VARIABLES FOR COLLABORATIVE MEASURES

The attempts in [1] and [5] to define variables for collaborative measures are quite different: [1] uses

$$(f_j)_{j=1,\ldots,q}, \tag{3}$$

where f_j denotes the number of papers with j authors. [5] uses

$$(x_{i,i'})_{i,i'=1,\ldots,n}, \tag{4}$$

where $x_{i,i'}$ denotes the number of times that author i and author i' have co-authored a paper.

Both approaches look very different and indeed they are: The one does not imply the other and vice versa, as the next two examples show.

Example II.1

Consider two collaborative situations, as shown in Fig. 3a and b ($n = 3$). In both cases we have $x_{1,2} = x_{2,1} = 2$, $x_{1,3} = x_{3,1} = 2$, $x_{2,3} = x_{3,2} = 2$. But in the first case $f_3 = 1$ and $f_2 = 3$ (all the other $f_j = 0$), while in the second case $f_3 = 2$ and all the other $f_j = 0$. Thus, equal values for $(x_{i,i'})_{i,i'=1,\ldots,n}$ do not imply equal values for $(f_j)_{j=1,\ldots,k}$.

Fig. 2. Another collaborative system.

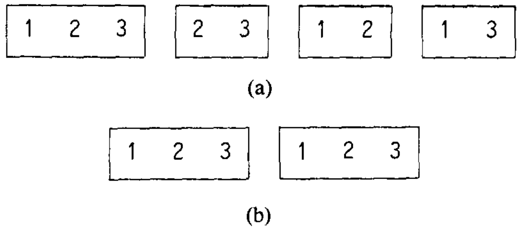

Fig. 3. Two collaborative situations.

Example II.2

Consider the two collaborative situations shown in Fig. 4a and b ($n = 3$). Here, in both cases $f_3 = 1, f_2 = 2$ (and all the other $f_j = 0$), but in case (a) we have $x_{1,2} = x_{2,1} = 2$, $x_{1,3} = x_{3,1} = 1$, and $x_{2,3} = x_{3,2} = 2$, while in case (b) we have: $x_{1,2} = x_{2,1} = 3$, $x_{1,3} = x_{3,1} = 1$, $x_{2,3} = x_{3,2} = 1$. Hence equal values of $(f_j)_{j=1,\ldots,q}$ do not imply equal values of $(x_{i,i'})_{i,i'=1,\ldots,n}$. So the approaches in [1] and [2] are not equivalent.

We can already conclude that, considering the above remark, neither of the attempts in [1] or [5] encompass all aspects of collaboration: the measures developed in terms of the variables $(f_j)_{j=1,\ldots,q}$ *or* the variables $(x_{i,i'})_{i,i'=1,\ldots,n}$ are not fine enough to distinguish all different collaborative situations. Nor are the two together adequate, as the next example shows.

Example II.3

Consider the two collaborative situations as in Fig. 5a and b ($n = 5$). Now, in both cases both the xs and the fs are equal. We now formulate the following definition.

Definition II.4

We define a *collaborative situation* over a set of objects as a family of subsets of these objects. Suppose we have two collaborative situations, each involving n objects $i = 1, \ldots, n$. We say that these two situations are *indistinguishable* if there exists a permutation π of the objects $1, \ldots, n$ such that, when π is applied to the first situation, we get the second one. An example is given in Fig. 6. The situation in Fig. 6a, when transformed via the permutation $\pi : \pi(1) = 3, \pi(2) = 2, \pi(3) = 4, \pi(4) = 1$ yields the situation in Fig. 6b.

It is clear that the collaborative structure of both situations in Fig. 6 is the same; in fact, this will be one of the principles for good collaboration measures: the values in indistinguishable situations must be the same (see section 3).

Now, do we have indistinguishable situations in Fig. 5? Certainly not; because if they were, the image of the box with elements 1 and 4 (occurring twice) should be two equal boxes of two elements in Fig. 5b. But all the boxes with two elements in Fig. 5b are different. Thus we have here two different (i.e., distinguishable) situations. This is also clear intuitively: in Fig. 5a, the object 5 appears only in one box, namely [1 2 3 5], while in Fig. 5b, all the elements in the box [1 2 3 4] appear at least once more in one of the boxes with two elements — a different collaborative situation indeed. So we conclude that

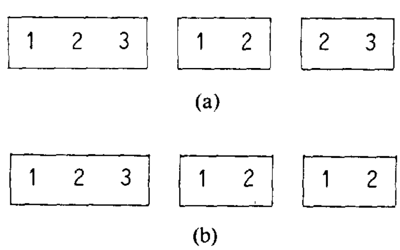

Fig. 4. Two collaborative situations.

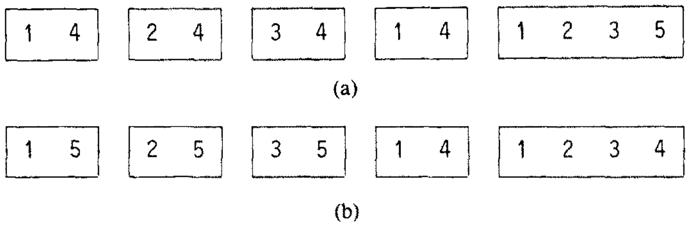

Fig. 5. Two collaborative situations ($n = 5$).

- definition II.4 makes it clear which collaborative situations are the same and which are not,
- the variables $(f_j)_{j=1,\ldots,q}$ *and* $(x_{i,i'})_{i,i'=1,\ldots,n}$ together do not suffice to describe a collaborative situation completely (cf. Fig. 5).

In this paper we study the possibility of describing a collaborative situation using only the indices i, i' and j, where, as above, $i, i' = 1, \ldots, n$, $j = 1, \ldots, q$.

The finest variables (i.e., distinguishing between as many distinguishable collaborative situations as possible) are the variables

$$(x_{i,i'}^{(j)})_{i,i'=1,\ldots,n;j=1,\ldots,q}, \tag{5}$$

where $x_{i,i'}^{(j)}$ denotes the number of times objects i and i' are together in a box with j items. The numbers $(x_{i,i'}^{(j)})$ are a reinforcement of the numbers (f_j) as well as of the numbers $(x_{i,i'})$.

PROPOSITION II.5

Given a system of numbers $(x_{i,i'}^{(j)})_{i,i'=1,\ldots,n;j=1,\ldots,q}$, *then we know the systems of numbers* $(x_{i,i'})_{i,i'=1,\ldots,n}$ *and* $(f_j)_{j=1,\ldots,q}$.

Proof. The proof follows from the obvious relations

$$x_{i,i'} = \sum_{j=1}^{q} x_{i,i'}^{(j)} \tag{6}$$

and

$$f_j = \frac{1}{j(j-1)} \sum_{\substack{i,i'=1 \\ i' \neq 1}}^{n} x_{i,i'}^{(j)}, \tag{7a}$$

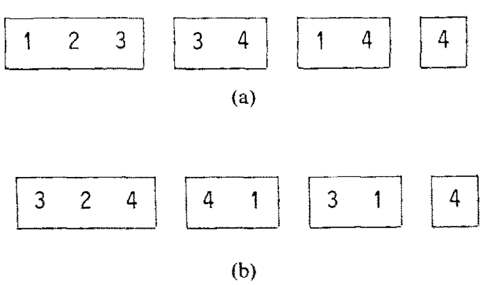

Fig. 6. Two collaborative situations (n objects $i = 1, \ldots, n$).

for $j \neq 1$ and

$$f_1 = \sum_{i=1}^{n} x_{i,i}^{(1)}. \tag{7b}$$

□

Although the variables $(x_{i,i'}^{(j)})$ are very detailed, they are not distinguishing between every two distinguishable collaborative situations. This is shown by the next example, constructed by H. Englisch:

Example II.6

The first collaborative situation consists of five times the boxes in Fig. 7. The second collaborative situation consists of *all* combinations of seven elements in boxes of three objects. Now, in both situations, *all* $x_{i,i'}^{(3)} = 5$ $(i \neq i')$, $x_{i,i}^{(3)} = 15$ and *all* $x_{i,i'}^{(j)} = 0$ $(j \neq 3)$. But both situations are distinguishable because the first situation contains the same boxes, whereas the second does not.

It is, however, true that, when only using the indices i, i', j, one cannot have finer variables than $x_{i,i'}^{(j)}$. To cover all distinguishable situations would require variables of the form $x_{i_1, i_2, \ldots, i_l}$, denoting the number of times objects $i_1, i_2, \ldots, i_l$ are together in one box ($i_1, \ldots, i_l$ all different and $l = 2, \ldots, n$). These intricate variables are not studied in this article.

Now that we have a set of variables convenient for describing a collaborative situation, we can turn our attention to functions of these variables that can serve the role of collaborative measures. But first we determine "good" principles that collaborative measures must satisfy.

III. COLLABORATIVE PRINCIPLES

Definition III.1

A collaborative measure is a function g of the variables $(x_{i,i'}^{(j)})_{i,i'=1,\ldots,n; j=1,\ldots,q}$ where $x_{i,i'}^{(j)}$ denotes the number of times that objects i and i' are together in a box that contains j objects. $x_{i,i'}^{(j)} = x_{i',i}^{(j)}$ for every i, i', and j; and q is the maximal number of objects in a single box.

Examples will follow in the next section. Note however that the function CC in [1], which we mentioned already in section 1, is of the above form:

$$CC = 1 - \frac{\sum_{j=1}^{q} \frac{1}{j} f_j}{N} = 1 - \frac{\sum_{j=1}^{q} \frac{1}{j} f_j}{\sum_{j=1}^{q} f_j}. \tag{1}$$

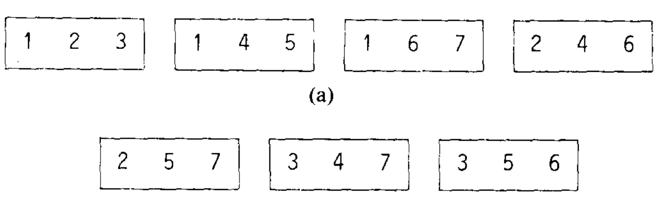

Fig. 7. Example II.6 (first collaborative situation).

Based on this,

$$CC = 1 - \frac{\sum_{ii',j} \frac{1}{j^2(j-1)} x_{i,i'}^{(j)} + \sum_{i=1}^{n} x_{i,i}^{(1)}}{\sum_{i,i',j} \frac{1}{j(j-1)} x_{i,i'}^{(j)} + \sum_{i=1}^{n} x_{i,i}^{(1)}} \tag{8}$$

where $\sum_{i,i',j}$ denotes $\sum_{i,i'=1,i\neq i'}^{n} \sum_{j=2}^{q}$. Similarly, the function s in [5] is also of this form:

$$s = \left(\sum_{\substack{i,i'=1 \\ i\neq i'}}^{n} \sqrt{x_{i,i'}} \right)^2 \tag{2}$$

$$s = \left(\sum_{\substack{i,i'=1 \\ i\neq i'}}^{n} \sqrt{\sum_{j=2}^{q} x_{i,i'}^{(j)}} \right)^2 \tag{9}$$

using (7) and (6), respectively. Thus CC and s satisfy our criterion of being a collaborative measure.

III.2. PRINCIPLE (P_1)

$$\text{If all } x_{i,i'}^{(j)} = 0 \text{ for } j = 2, \ldots, q, \text{ and } i \neq i', \text{ then } g(x_{i,i'}^{(j)}) = 0. \tag{10}$$

Here we use the more compact notation

$$g(x_{i,i'}^{(j)}) = g(x_{i,i'}^{(j)}; i, i' = 1, \ldots, n; j = 1, \ldots, q).$$

This means that if there are only boxes with one object (e.g., there are only single-authored papers) then $g = 0$.

III.3. PRINCIPLE (P_2)

$$\text{If there is "maximal collaboration," then } g = 1. \tag{11}$$

What do we mean by maximal collaboration? In collaboration studies we do not measure the total number of boxes (e.g., papers); the latter aspect is more an aspect of productivity. By maximal collaboration, we mean, given $n > 1$ (the number of objects) and N (the number of boxes—fixed but arbitrary), we have that all n objects are in all boxes. Hence we mean a situation as in Fig. 8. Based on (P_1) and (P_2), it is logical to require

III.4. PRINCIPLE (P_3)

$$0 \leqq g \leqq 1 \tag{12}$$

always.

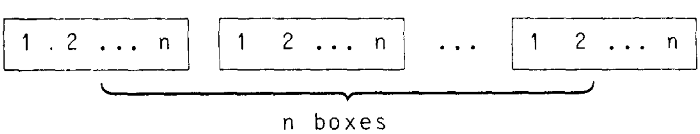

Fig. 8. Maximal collaboration.

III.5. Principle (P_4)

"Scale invariance": Let $(x_{i,i'}^{(j)})$ and $(y_{i,i'}^{(j)})$ represent two existing collaborative situations such that there exists $a \in \mathbf{N}$ such that

$$y_{i,i'}^{(j)} = ax_{i,i'}^{(j)}, \tag{13}$$

for every $i, i' = 1, \ldots, n$ and $j = 1, \ldots, q$, then

$$g(y_{i,i'}^{(j)}) = g(x_{i,i'}^{(j)}). \tag{14}$$

Again, this condition is introduced because we do not wish to measure "productivity," and, apart from productivity, the collaborative patterns (13) are the same.

III.6. Principle (P_5)

The following property was already discussed in definition II.4: If we have two collaborative situations that are indistinguishable and suppose, hence, that π is the permutation of $\{1, \ldots, n\}$ that transforms the first situation into the second, then

$$g(x_{i,i'}^{(j)}) = g(x_{\pi(i),\pi(i')}^{(j)}) \tag{15}$$

for all $i, i' = 1, \ldots, n$; $j = 1, \ldots, q$.

On the other hand, if two situations are not indistinguishable, we will prefer that the collaborative measure tends to give different values for different situations. The next principles expand this idea.

III.7. Principle (P_6)

If more boxes with one object are added to a situation where boxes with more than one object exist, g decreases strictly. This is reasonable; if two collaborative situations are the same except that in the second there are more single-authored papers, then g (of the second situation) is strictly smaller than g (of the first situation).

We now define an important principle that deals with the fine structure of collaborative situations.

III.8. Principle (P_7)

"Strict Concavity." We start by recalling the definition of the insufficient, but better known property of "concavity."

Let $v: \mathbf{R}^p \to \mathbf{R}$ be any real-valued function, defined on the p-dimensional space $\mathbf{R}^p$ (p fixed). Let us denote $\mathbf{X}, \mathbf{Y} \in \mathbf{R}^p$ for $\mathbf{X} = (x_1, \ldots, x_p)$, $\mathbf{Y} = (y_1, \ldots, y_p)$. We say that the function v is *concave* if, for every $\mathbf{X}, \mathbf{Y} \in \mathbf{R}^p$ and every $\lambda \in [0,1]$,

$$v(\lambda \mathbf{X} + (1-\lambda)\mathbf{Y}) \geqq \lambda v(\mathbf{X}) + (1-\lambda) v(\mathbf{Y}). \tag{16}$$

This means that the value of v in a point on the line segment between two points is at least as much as the same linear combination of the values of v in the two points. For $p = 1$ it is easy to depict this situation (see Fig. 9).

Why is this a property that collaborative measures should fulfill? The property expresses the fact that the collaborative value of a situation that is a (weighted) average of two collaborative situations is higher than the (weighted) average of the respective collaborative values. In other words, the more equal collaboration links there are between the members of a group, the higher collaborative value.

Note: Concavity (together with (P_5)) also implies properties such as, for every $(x_1, \ldots, x_p) \in \mathbf{R}^p$:

$$v\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \ldots, x_p\right) \geqq v(x_1, x_2, \ldots, x_p)$$

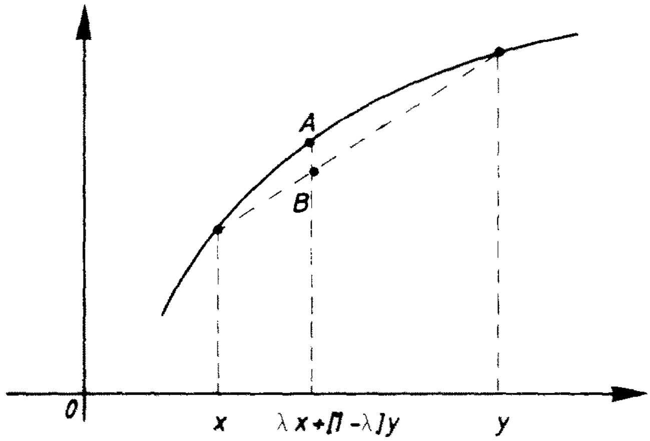

Fig. 9. The ordinate of A is $v(\lambda x + (1-\lambda)y)$. The ordinate of B is $\lambda v(x) + (1-\lambda)v(y)$.

(cf. inequality (3) in [5]), which again expresses the fact that more equal collaborative links give higher collaborative values. To prove the above inequality, just apply concavity and (P_5) to the vectors $\mathbf{X} = (x_1,x_2,\ldots,x_n)$, $\mathbf{Y} = (x_2,x_1,\ldots,x_n)$ and $\lambda = 0.5$.

Of course, in unequal situations, a strict inequality in (16) would be preferred. But this can only be true if **X** and **Y** themselves represent different situations. Indeed, if **X** and **Y** represent collaborative situations with equal collaboration strength (e.g., $\mathbf{Y} = a\mathbf{X}$ for a certain $a \in \mathbf{N}$ (cf. section III.5)), then $\mathbf{X} + (1-\lambda)\mathbf{Y}$ is also a situation with equal collaboration strength. In fact, we can prove: If there is an $a \in \mathbf{N}$ such that $\mathbf{Y} = a\mathbf{X}$, then for every $\lambda \in [0,1]$:

$$\begin{aligned} v(\lambda\mathbf{X} + (1-\lambda)\mathbf{Y}) &= v(\lambda\mathbf{X} + (1-\lambda)a\mathbf{X}) \\ &= v((\lambda + (1-\lambda)a)\mathbf{X}). \end{aligned}$$

Since v must satisfy principle (P_4), we conclude

$$\begin{aligned} v(\lambda\mathbf{X} + (1-\lambda)\mathbf{Y}) &= v((\lambda + (1-\lambda)a)\mathbf{X}) \\ &= v(\mathbf{X}) \\ &= \lambda v(\mathbf{X}) + (1-\lambda)v(\mathbf{X}) \\ &= \lambda v(\mathbf{X}) + (1-\lambda)v(\mathbf{Y}). \end{aligned}$$

Of course, the same is true for every $\mathbf{X},\mathbf{Y} \in \mathbf{R}^p$ with $\lambda = 0$ or 1.

Thus the strongest possible definition of "strict concavity" is as follows. A function $v : \mathbf{R}^p \to \mathbf{R}$ is *strictly concave* if it is concave and if, for every $\lambda \in]0,1[$ (the open interval from 0 to 1) and every $\mathbf{X},\mathbf{Y} \in \mathbf{R}^p$, for which there is no $a \in \mathbf{R}^+$ such that $\mathbf{Y} = a\mathbf{X}$, we have

$$v(\lambda\mathbf{X} + (1-\lambda)\mathbf{Y}) > \lambda v(\mathbf{X}) + (1-\lambda)v(\mathbf{Y}). \tag{17}$$

This is the content of principle (P_7), to be checked for every candidate collaboration measure g. Note that here the maximal dimension

$$p = \sum_{j=1}^{q} j^2 f_j.$$

Note 1: As requested by one of the referees, we give an example of the purpose of the strict concavity principle. Suppose we have two collaborative situations as in Figs. 10a and 10b ($n = 3$). The average ($\lambda = 0.5$) of these situations is the collaborative situation in

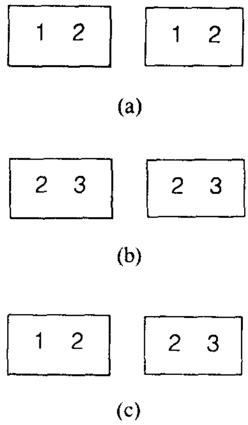

Fig. 10. Two collaborative situations ($n = 3$) and their average (10c).

Fig. 10c. It is clear that the collaborative value of Fig. 10c is much higher than the average of the (in fact equal) collaborative values of Figs. 10a and 10b. Any collaborative measure that satisfies (P_7) will fulfill this requirement. Many other examples in the line of the above show that (P_7) is an important property.

Note 2: As remarked to me by H. Englisch, principle (P_7) is not always "perfect." This is, however, linked with the imperfection of the variables $x_{i,i'}^{(j)}$ (cf. example II.6). Returning to this example, we feel (with Englisch) that the collaborative value of the first collaborative situation is strictly less than the collaborative value of the second situation; yet, because the $(x_{i,i'}^{(j)})$ are the same for both situations, the values of any collaborative measure studied here will be the same. We can even obtain the reverse inequality (as communicated to me by Englisch): take the first situation in example II.6 as our first collaborative situation. As our second collaborative situation we take the second situation in example II.6, but changed as follows:

replace box [5 6 7] by one more box [1 2 3].

It is clear that the collaborative value of situation 2 is higher than the one of situation 1; yet, due to strict concavity, any measure satisfying (P_7) will give oppositely ranked values. We leave it as an open problem to give a "perfect" statement of principle (P_7).

The last "natural" collaborative principle is a property that could be called the "Bridging Principle."

III.9. Principle (P_8)

Suppose we have two identical collaborative situations, described by the numbers $(x_{i,i'}^{(j)})$. Suppose we add one box (e.g., one more paper) with two objects to each situation. Suppose that in situation 1, the objects in the new box appeared already together in at least one of the other boxes, while in situation 2 the two objects in the new box are coupled for the first time. Then the collaboration value of situation 2 is strictly higher than the one for situation 1. An example makes this principle clear:

To the common situation depicted in Fig. 11a we add,
in the first case [1 2] and in the second case [3 4],

yielding the two situations shown in Figs. 11b and 11c. Principle (P_8) now requires that the collaborative value of Fig. 11b be strictly smaller than the one of Fig. 11c.

This ends our survey of "natural" collaborative principles. By stating them we are not assured of the existence of measures that satisfy these principles. In the next section we will investigate a few of the known ones and introduce new ones.

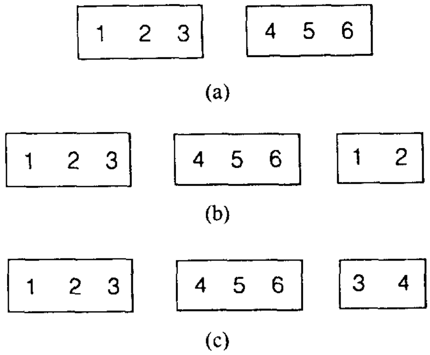

Fig. 11. Illustration of the "Bridging Principle."

IV. COLLABORATIVE MEASURES

We will study the following possible functions, defined in section I:

$$CC = 1 - \frac{\sum_{j=1}^{q} \frac{1}{j} f_j}{N}, \tag{1}$$

where $N = \sum_{j=1}^{q} f_j$ and

$$s = \left(\sum_{\substack{i,i'=1 \\ i \neq i'}}^{n} \sqrt{x_{i,i'}} \right)^2. \tag{2}$$

An evident additional measure to study is

$$f = \sum_{i,i',j} x_{i,i'}^{(j)}, \tag{19}$$

where $\sum_{i,i',j}$ denotes $\sum_{i,i'=1,i\neq i'}^{n} \sum_{j=2}^{q}$.

Inspired by (2), but preferring the variables $x_{i,i'}^{(j)}$ (cf. section II), we introduce

$$h = \left(\sum_{i,i',j} \sqrt{x_{i,i'}^{(j)}} \right)^2. \tag{20}$$

These will not be the final form of the functions; when necessary for normalization (mainly because of principle (P_2)), we will need to multiply these functions by a fixed factor. It is our purpose to develop good collaborative measures in a logical way. At the end of the paper we hope to have presented the final forms of good collaborative measures.

IV.1. *The collaborative coefficient CC*

Formula (1) implies that, if we have only single-authored papers, then $CC = 0$. However, CC for the maximal situation as described in (P_2) is

$$CC_{\max} = 1 - \frac{N \cdot \frac{1}{n}}{N} = 1 - \frac{1}{n} \neq 1.$$

Consequently, we redefine

$$CC^* = \frac{n}{n-1} CC = \frac{n}{n-1} \left[1 - \frac{\sum_{j=1}^{q} \frac{1}{j} f_j}{\sum_{j=1}^{q} f_j} \right] \tag{21}$$

so that CC^* will satisfy (P_2) (and also (P_1), of course) as well as (P_3). We will henceforth work with CC^*.

For (P_4), consider two existing collaborative situations, represented by $(x_{i,i'}^{(j)})$ and $(y_{i,i'}^{(j)})$, such that there is $a \in \mathbf{N}$ for which

$$y_{i,i'}^{(j)} = ax_{i,i'}^{(j)}.$$

This implies that $f_j^{(2)}$, the (f_j) of the second system, relates to $f_j^{(1)}$, the (f_j) of the first system, as

$$f_j^{(2)} = af_j^{(1)} \tag{22}$$

for every $j = 1, \ldots, k$.

Indeed, according to (7a) and (7b) one has, for every $j \geqq 2$,

$$f_j^{(2)} = \frac{1}{j(j-1)} \sum_{\substack{i,i'=1 \\ i \neq i'}}^{n} y_{i,i'}^{(j)}$$

$$f_j^{(2)} = \frac{1}{j(j-1)} \sum_{\substack{i,i'=1 \\ i \neq i'}}^{n} ax_{i,i'}^{(j)}$$

$$f_j^{(2)} = af_j^{(1)},$$

and the same if $j = 1$. Consequently, denoting CC_i^* for CC^* in the ith situation, $i = 1,2$,

$$CC_2^* = \frac{n}{n-1}\left[1 - \frac{\sum_{j=1}^{q} \frac{1}{j} f_j^{(2)}}{\sum_{j=1}^{q} f_j^{(2)}}\right] = \frac{n}{n-1}\left[1 - \frac{\sum_{j=1}^{q} \frac{1}{j} af_j^{(1)}}{\sum_{j=1}^{q} af_j^{(1)}}\right] = CC_1^*.$$

Hence (P_4) is satisfied.

Principle (P_5) is trivially satisfied. We will now prove that CC^* satisfies (P_6). Hence suppose that

$$CC_1^* = 1 - \frac{\sum_{j=1}^{q} \frac{1}{j} f_j}{N}$$

and

$$CC_2^* = 1 - \frac{\sum_{j=1}^{q} \frac{1}{j} f_j'}{N'},$$

where $N' - N = f_1' - f_1 > 0$ (and hence $f_j = f_j'$ for every $j = 2, \ldots, q$) and also at least one $f_j > 0$ $(j = 2, \ldots, q)$. Now

$$\begin{aligned} CC_2^* - CC_1^* &= -\frac{\sum_{j=2}^{q} \frac{1}{j} f_j'}{N'} - \frac{f_1'}{N'} + \frac{\sum_{j=2}^{q} \frac{1}{j} f_j}{N} + \frac{f_1}{N} \\ &= \left(\frac{f_1}{N} - \frac{f_1'}{N'}\right) + \sum_{j=2}^{q} \frac{1}{j} f_j \left(\frac{1}{N} - \frac{1}{N'}\right) \\ &= \left(\frac{f_1}{N} - \frac{f_1 + N' - N}{N'}\right) + \sum_{j=2}^{q} \frac{1}{j} f_j \left(\frac{1}{N} - \frac{1}{N'}\right) \\ &= -\frac{N' - N}{N'} + \sum_{j=1}^{q} \frac{1}{j} f_j \left(\frac{N' - N}{NN'}\right) \\ &< 0 \end{aligned}$$

if and only if

$$\left(\sum_{j=1}^{q} \frac{1}{j} f_j\right) \cdot \frac{1}{N} - 1 < 0.$$

Hence, this is true if and only if

$$\sum_{j=1}^{q} \frac{1}{j} f_j < N.$$

This is clearly true since

$$\sum_{j=1}^{q} f_j = N$$

and since at least one $f_j > 0$ $(j = 2, \ldots, q)$. This proves that CC^* satisfies (P_6).

We now turn our attention to the principle (P_7). That CC^* is concave is clear:

$$CC^* = \frac{n}{n-1}\left[1 - \frac{\displaystyle\sum_{i,i',j} \frac{1}{j^2(j-1)} x_{i,i'}^{(j)} + \sum_{i=1}^{n} x_{i,i}^{(1)}}{N}\right],$$

using (1), (7a) and (21) (N, n and k are fixed here). So CC^* is a linear function of $x_{i,i'}^{(j)}$ and hence concave. But the linearity obviously implies that CC^* is *not* strictly concave. So an important aspect is lacking here. Finally, principle (P_8) is also not satisfied since both situations keep the same number of boxes, and hence CC^* cannot distinguish between the two situations as described in (P_8).

IV.2. *The Englisch measure* s

In [5], Englisch defines more general functions than function (2), but this generality does not yield more collaborative properties; so we restrict our attention to the simpler form of s:

$$s = \left(\sum_{\substack{i,i'=1 \\ i\neq i'}}^{n} \sqrt{x_{i,i'}}\right)^2.$$

Principle (P_1) is obviously satisfied. For the "maximal cooperation" as described in (P_2) we now find

$$s_{\max} = \left(n(n-1)\sqrt{N}\right)^2 = n^2(n-1)^2N.$$

Consequently, we define

$$s^* = \frac{1}{n^2(n-1)^2N}\left(\sum_{\substack{i,i'=1 \\ i\neq i'}}^{n} \sqrt{x_{i,i'}}\right)^2.$$

Written in terms of $x_{i,i'}^{(j)}$ this yields, by (6),

$$s^* = \frac{1}{n^2(n-1)^2N}\left(\sum_{\substack{i,i'=1 \\ i\neq i'}}^{n} \sqrt{\sum_{j=1}^{q} x_{i,i'}^{(j)}}\right)^2. \tag{23}$$

We will henceforth work with s^*. Clearly (P_1), (P_2), and (P_3) are satisfied by s^*. If, for (P_4), $(x_{i,i'}^{(j)})$ and $(y_{i,i'}^{(j)})$ represent two existing collaborative situations such that there exists $a \in \mathbf{N}$ for which

$$y_{i,i'}^{(j)} = ax_{i,i'}^{(j)}$$

for every $i, i' = 1, \ldots, n$ and $j = 1, \ldots, q$, then N, the number of boxes, is also multiplied by a. Hence, for the $(y_{i,i'}^{(j)})$ case we have

$$s_2^* = \frac{1}{n^2(n-1)^2 aN} \left(\sum_{\substack{i,i'=1 \\ i \neq i'}}^{n} \sqrt{\sum_{j=1}^{q} a x_{i,i'}^{(j)}} \right)^2 = s_1^*,$$

with the as cancelling.

Clearly (P_5) is true. (P_6) is also true since adding papers with a single author increases N but leaves the $x_{i,i'}$ invariant. Hence s^* decreases (strictly, if at least one $x_{i,i'}^{(j)} \neq 0$). This brings us to the study of principle (P_7). We first show that s^* is concave. In order to simplify the notation we will denote $(x_{i,i'}^{(j)})_{i,i'=1,\ldots,n; i \neq i'; j=2,\ldots,q}$ by $\mathbf{X} = (x_l^{(j)})_{l=1,\ldots,m; j=2,\ldots,q}$ and similarly for the $(y_{i,i'}^{(j)})$. We furthermore denote $C = 1/[n^2(n-1)^2 N]$. In this notation

$$s^*(\mathbf{X}) = C\left(\sum_{l=1}^{m} \sqrt{\sum_{j=2}^{q} x_l^{(j)}} \right)^2.$$

Let $\lambda \in [0,1]$. Then

$$\begin{aligned}
s^*(\lambda\mathbf{X} + (1-\lambda)\mathbf{Y}) &= C\left(\sum_{l} \sqrt{\sum_{j} (\lambda x_l^{(j)} + (1-\lambda) y_l^{(j)})} \right)^2 \\
&= C\Bigg[\sum_{l} \left(\sum_{j} (\lambda x_l^{(j)} + (1-\lambda) y_l^{(j)}) \right) \\
&\quad + \sum_{l \neq l'} \sqrt{\left(\sum_{j} (\lambda x_l^{(j)} + (1-\lambda) y_l^{(j)}) \right)\left(\sum_{j} (\lambda x_{l'}^{(j)} + (1-\lambda) y_{l'}^{(j)}) \right)} \Bigg] \\
&= \lambda C\left(\sum_{l} \left(\sum_{j} x_l^{(j)} \right) \right) + (1-\lambda) C\left(\sum_{l} \left(\sum_{j} y_l^{(j)} \right) \right) \\
&\quad + C \sum_{l \neq l'} \sqrt{\left(\sum_{j} (\lambda x_l^{(j)} + (1-\lambda) y_l^{(j)}) \right)\left(\sum_{j} (\lambda x_{l'}^{(j)} + (1-\lambda) y_{l'}^{(j)}) \right)}.
\end{aligned}$$

So the proof is finished if we can show that

$$\begin{aligned}
&\sum_{l \neq l'} \sqrt{\left(\sum_{j} (\lambda x_l^{(j)} + (1-\lambda) y_l^{(j)}) \right)\left(\sum_{j} (\lambda x_{l'}^{(j)} + (1-\lambda) y_{l'}^{(j)}) \right)} \\
&\quad \geq \lambda \sum_{l \neq l'} \sqrt{\left(\sum_{j} x_l^{(j)} \right)\left(\sum_{j} x_{l'}^{(j)} \right)} + (1-\lambda) \sum_{l \neq l'} \sqrt{\left(\sum_{j} y_l^{(j)} \right)\left(\sum_{j} y_{l'}^{(j)} \right)}.
\end{aligned}$$

For this it is sufficient to show that, for every $l \neq l'$,

$$\begin{aligned}
&\sqrt{\left(\sum_{j} (\lambda x_l^{(j)} + (1-\lambda) y_l^{(j)}) \right)\left(\sum_{j} (\lambda x_{l'}^{(j)} + (1-\lambda) y_{l'}^{(j)}) \right)} \\
&\quad \geqq \lambda \sqrt{\left(\sum_{j} x_l^{(j)} \right)\left(\sum_{j} x_{l'}^{(j)} \right)} + (1-\lambda) \sqrt{\left(\sum_{j} y_l^{(j)} \right)\left(\sum_{j} y_{l'}^{(j)} \right)}.
\end{aligned}$$

If we take the square of both sides and simplify, we find (if $\lambda \neq 0$ or 1; if $\lambda = 0$ or 1, the concavity is trivial)

$$\begin{aligned}
&\left(\sum_{j} x_l^{(j)} \right)\left(\sum_{j} y_{l'}^{j} \right) + \left(\sum_{j} y_l^{(j)} \right)\left(\sum_{j} x_{l'}^{(j)} \right) \\
&\quad \geqq 2\sqrt{\left(\sum_{j} x_l^{(j)} \right)\left(\sum_{j} x_{l'}^{(j)} \right)\left(\sum_{j} y_l^{(j)} \right)\left(\sum_{j} y_{l'}^{(j)} \right)}
\end{aligned}$$

or, equivalently, for every $l \neq l'$,

$$\left(\sqrt{\left(\sum_j x_l^{(j)}\right)\left(\sum_j y_{l'}^{(j)}\right)} - \sqrt{\left(\sum_j y_l^{(j)}\right)\left(\sum_j x_{l'}^{(j)}\right)}\right)^2 \geqq 0, \tag{24}$$

which is always true. This shows that s^* is concave. Formula (24) can also be used to test for strict concavity; for at least one couple (l, l') with $l \neq l'$ one should find (24), but with $\geqq$ replaced by $>$. Thus, s^* will not be strictly concave if there exist any vectors $\mathbf{X}$ and $\mathbf{Y}$ for which there is no $a > 0$ such that $\mathbf{Y} = a\mathbf{X}$, and such that, for every $l \neq l'$,

$$\left(\sum_j x_l^{(j)}\right)\left(\sum_j y_{l'}^{(j)}\right) = \left(\sum_j y_l^{(j)}\right)\left(\sum_j x_{l'}^{(j)}\right). \tag{25}$$

Denoting $\sum_j x_l^{(j)}$ by u_l and $\sum_j y_l^{(j)}$ by v_l, we have the condition

$$u_l v_{l'} = v_l u_{l'} \tag{26}$$

for every $l \neq l'$, $l, l' = 1, \ldots, n$. Hence

$$\begin{cases} u_1 v_2 = v_1 u_2 \\ u_1 v_3 = v_1 u_3 \\ \vdots \qquad \vdots \\ u_1 v_n = v_1 u_n. \end{cases}$$

Consequently,

$$\frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3} = \ldots = \frac{u_n}{v_n}. \tag{27}$$

So (27) is equivalent with: there exists $b > 0$ such that, for every $l = 1, \ldots, n$,

$$u_l = b v_l;$$

hence

$$\sum_j x_l^{(j)} = b \sum_j y_l^{(j)}$$

or, in the detailed notation:

$$\sum_j x_{i,i'}^{(j)} = b \sum_j y_{i,i'}^{(j)} \tag{30}$$

for all $i \neq i'$, $i, i' = 1, \ldots, n$. We will now present an example of vectors $\mathbf{X}$ and $\mathbf{Y}$ for which there is no $a > 0$ such that $\mathbf{Y} = a\mathbf{X}$ but for which (31) is valid for even $b = 1$, hence showing that s^* is *not* strictly concave. The example is contained in Fig. 12 ($n = 4$). We have here:

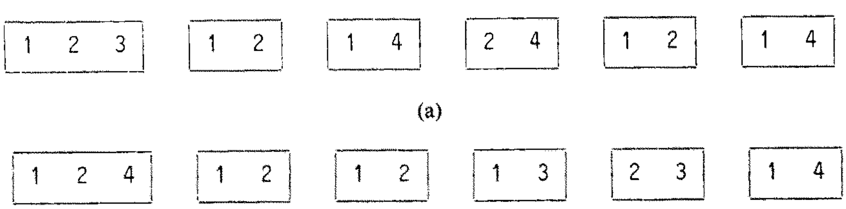

Fig. 12. An example showing that s^* is not strictly concave.

$$\sum_j x_{1,2}^{(j)} = 3 = \sum_j y_{1,2}^{(j)};\quad \sum_j x_{1,3}^{(j)} = 1 = \sum_j y_{1,3}^{(j)};$$

$$\sum_j x_{1,4}^{(j)} = 2 = \sum_j y_{1,4}^{(j)};\quad \sum_j x_{2,3}^{(j)} = 1 = \sum_j y_{2,3}^{(j)};$$

$$\sum_j x_{2,4}^{(j)} = 1 = \sum_j y_{2,4}^{(j)};\quad \sum_j x_{3,4}^{(j)} = 0 = \sum_j y_{3,4}^{(j)}.$$

But $\mathbf{X} \neq a\mathbf{Y}$ for any a, where $\mathbf{X} = (x_{i,i'}^{(j)})$, $\mathbf{Y} = (y_{i,i'}^{(j)})$, since (only denoting the coordinates with $i \neq i'$)

$$\mathbf{X} = (x_{1,2}^{(2)}, x_{1,3}^{(2)}, x_{1,4}^{(2)}, x_{2,3}^{(2)}, x_{2,4}^{(2)}, x_{3,4}^{(2)}, x_{1,2}^{(3)}, x_{1,3}^{(3)}, x_{1,4}^{(3)}, x_{2,3}^{(3)}, x_{2,4}^{(3)}, x_{3,4}^{(3)})$$

$$= (2, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 0)$$

while

$$\mathbf{Y} = (y_{1,2}^{(2)}, y_{1,3}^{(2)}, y_{1,4}^{(2)}, y_{2,3}^{(2)}, y_{2,4}^{(2)}, y_{3,4}^{(2)}, y_{1,2}^{(3)}, y_{1,3}^{(3)}, y_{1,4}^{(3)}, y_{2,3}^{(3)}, y_{2,4}^{(3)}, y_{3,4}^{(3)})$$

$$= (2, 1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0)$$

and furthermore, there is no permutation π of $\{1,2,3,4\}$ such that Fig. 12a is transformed into Fig. 12b: indeed, in Fig. 12a one has two times two equal boxes with two elements, while in Fig. 12b only the box $\boxed{1\ 2}$ appears twice. Hence s^* is not strictly concave. We finally check principle (P_8).

Let $(x_{i,i'}^{(j)})_{i,i'=1,\ldots,n;j=1,\ldots q}$ represent the common situation before the new box is added. Let $\boxed{i_1, i_1'}$ denote the new box added in the first case. Of course, $i_1, i_1' \in \{1, \ldots, n\}$ since in the first case i_1 and i_1' were already coupled. Let $\boxed{i_2, i_2'}$ be the new box added in the second case. Here we can also suppose that $i_2, i_2' \in \{1, \ldots, n\}$ (otherwise increase n and adapt $(x_{i,i'}^{(j)})$ of the common situation accordingly). Denote $C = 1/[n^2(n-1)^2N]$. Then s^* in the first, respectively the second, case is:

$$s_1^* = C\left(\sum_{\substack{i,i'=1\\ i\neq i'\\ i,i'\neq i_1,i_1'}}^{n} \sqrt{\sum_{j=2}^{q} x_{i,i'}^{(j)}} + 2\sqrt{\sum_{j=2}^{q} x_{i_1,i_1'}^{(j)} + 1}\right)$$

$$s_2^* = C\left(2 + \sum_{\substack{i,i'=1\\ i\neq i'}}^{n} \sqrt{\sum_{j=2}^{q} x_{i,i'}^{(j)}}\right).$$

Hence

$$\sqrt{\frac{s_2^*}{C}} - \sqrt{\frac{s_1^*}{C}} = 2 + 2\sqrt{\sum_{j=2}^{q} x_{i_1,i_1'}^{(j)}} - 2\sqrt{\sum_{j=2}^{q} x_{i_1,i_1'}^{(j)} + 1} > 0,$$

since, for every $\alpha > 0$, $1 + \sqrt{\alpha} > \sqrt{\alpha + 1}$ (and since $\sum_{j=2}^{q} x_{i_1,i_1'}^{(j)} > 0$, since the box $\boxed{i_1, i_1'}$ existed already in the common situation). Hence also $s_2^* > s_1^*$. So s^* satisfies principle (P_8).

IV.3. *The simple measure f*

We defined the measure f as

$$f = \sum_{i,i',j} x_{i,i'}^{(j)}.$$

We introduced f because of its simplicity. Principle (P_1) is obviously satisfied. Furthermore, in the maximal situation, we have

$$f_{\max} = n(N-1)N$$

so that again we define

$$f^* = \frac{\sum_{i,i',j} x_{i,i'}^{(j)}}{n(n-1)N}. \tag{29}$$

Hence, for f^*, (P_1), (P_2), as well as (P_3) are satisfied. (P_4) is also true, by the same argument as in section IV.2 and (P_5) and (P_6) are trivial. Since (29) is linear in $x_{i,i'}^{(j)}$ (since k, n and N are fixed), we have here a concave function that is *not* strictly concave. Hence (P_7) is not valid. Furthermore, principle (P_8) is not valid, since in (29), one simply adds the value $x_{i,i'}^{(j)}$.

IV.4. *The function h*

Certainly (P_1) is valid. For (P_2) we must again divide by

$$h_{\max} = n^2(n-1)^2N,$$

since this is the value of h in the case of Fig. 8 (cf. also section 4.2). Hence, we define

$$h' = \frac{1}{n^2(n-1)^2N}\left(\sum_{i,i',j} \sqrt{x_{i,i'}^{(j)}}\right)^2. \tag{30}$$

However, even h' is not always inferior to 1.

Example:

Consider the system in Fig. 13. Then $h' = 1.1657$. The reason we can find values above one (contrary to the case of s^*) is that the more boxes with different number j of objects we have, we add more terms in h' before the $\sqrt{\ }$-sign, then in the "maximal" case above, which only looks at boxes with the same number n of objects. Otherwise said, if we only have boxes with j objects (one j, fixed) then $h' \leqq 1$ always. We therefore define

$$K = \{j \mid j \geqq 2, f_j \neq 0\}$$

and $k = \#K$, the cardinality of the set K. Note that, for our calculations (cf. (19)),

$$\sum_{i,i',j} = \sum_{\substack{i,i'=1\\ i\neq i'}}^{n} \sum_{j=2}^{q} = \sum_{\substack{i,i'=1\\ i\neq i'}}^{n} \sum_{j\in K}.$$

We show now that dividing h' by k^2 gives a function

$$h^* = \frac{1}{n^2(n-1)^2Nk^2}\left(\sum_{i,i',j} \sqrt{x_{i,i'}^{(j)}}\right)^2 \tag{31}$$

such that in *all* cases: $h^* \leqq 1$. The proof goes as follows: By definition of K and k:

$$\left(\sum_{i,i',j} \sqrt{x_{i,i'}^{(j)}}\right)^2 = \left(\sum_{j\in K} \sum_{\substack{i,i'=1\\ i\neq i'}}^{n} \sqrt{x_{i,i'}^{(j)}}\right)^2$$

$$\leq k \sum_{j\in K} \left(\sum_{\substack{i,i'=1\\ i\neq i'}}^{n} \sqrt{x_{i,i'}^{(j)}}\right)^2,$$

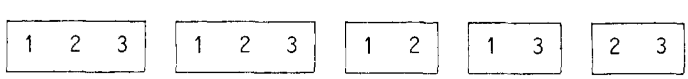

Fig. 13. A case where $h' > 1$.

by the Cauchy-Schwarz inequality (see, e.g., [2]) (take $(\sum_{i,i'=1,i\neq i'}^{n} \sqrt{x_{i,i'}^{(j)}})_{j\in K}$ and $(1,1,\ldots,1) \in \mathbf{R}^k$ as the two vectors).

$$h^* \leqq \frac{1}{n^2(n-1)^2Nk^2}\, k \sum_{j\in K} \Bigg(\sum_{\substack{i,i'=1\\ i\neq i'}}^{n} \sqrt{x_{i,i'}^{(j)}} \Bigg)^2. \tag{32}$$

Now, dealing with a fixed j, we have by the above reasoning (leading to (30)) that

$$\frac{1}{n^2(n-1)^2N} \Bigg(\sum_{\substack{i,i'=1\\ i\neq i'}}^{N} \sqrt{x_{i,i'}^{(j)}} \Bigg)^2 \leqq 1. \tag{33}$$

Hence, (32) and (33) imply

$$h^* \leqq \frac{k \sum_{j\in K} 1}{k^2} = 1.$$

As will be seen in proposition IV.7.2, the division of h' by k^2 is also necessary for reasons of comparison with s^*.

Obviously for h^*, principles (P_1), (P_2), and (P_3) are satisfied. The validity of (P_4) is proved as in section IV.2. (P_5) is obvious and (P_6) too (since N (and possibly k) are the only parameters that change in (30): they increase, hence h^* decreases if at least one $x_{i,i'}^{(j)} \neq 0$). h^* is strictly concave. The proof is similar to that of section IV.2; therefore we omit it.

However, h^* does not satisfy principle (P_8). Consider the example in Fig. 11b and 11c. In *both* cases $h^* = 0.0181$, contradicting the bridging principle.

IV.5. *A measure that satisfies all principles* (P_1) *through* (P_8)

As is clear from Table 1, no measure studied above satisfies all principles (P_1) through (P_8). Especially the measures CC^* and f^* lack the "fine" principles (P_7) and (P_8). The functions s^* and h^* are better since they satisfy seven principles. However, from the above, it is easy to construct a measure that satisfies *all* principles (P_1) through (P_8), namely $\gamma^* = (s^* + h^*)/2$.

THEOREM

The function $\gamma^* = (s^* + h^*)/2$ satisfies the principles (P_1) through (P_8).

Table 1. Results of discussed situations

1) Fig.	1	$CC^*=0.3714$,	$\sqrt[4]{s^*}=0.3694$,	$\sqrt{f^*}=0.2390$,	$\sqrt[4]{h^*}=0.2245$,	$\gamma_1=0.3206$,	$\gamma_2=0.2970$
2) Fig.	2	$CC^*=0.2000$,	$\sqrt[4]{s^*}=0.2582$,	$\sqrt{f^*}=0.1491$,	$\sqrt[4]{h^*}=0.1826$,	$\gamma_1=0.2295$,	$\gamma_2=0.2204$
3) Fig.	3a	$CC^*=0.8125$,	$\sqrt[4]{s^*}=0.8409$,	$\sqrt{f^*}=0.7071$,	$\sqrt[4]{h^*}=0.7071$,	$\gamma_1=0.7825$,	$\gamma_2=0.7740$
	3b	$CC^*=1.000$,	$\sqrt[4]{s^*}=1.000$,	$\sqrt{f^*}=1.000$,	$\sqrt[4]{h^*}=1.000$,	$\gamma_1=1.000$,	$\gamma_2=1.000$
4) Fig.	4a	$CC^*=0.8333$,	$\sqrt[4]{s^*}=0.8584$,	$\sqrt{f^*}=0.7454$,	$\sqrt[4]{h^*}=0.6936$,	$\gamma_1=0.7888$,	$\gamma_2=0.7760$
	4b	$CC^*=0.8333$,	$\sqrt[4]{s^*}=0.8475$,	$\sqrt{f^*}=0.7454$,	$\sqrt[4]{h^*}=0.6517$,	$\gamma_1=0.7681$,	$\gamma_2=0.7496$
5) Fig.	5a	$CC^*=0.6875$,	$\sqrt[4]{s^*}=0.6489$,	$\sqrt{f^*}=0.4472$,	$\sqrt[4]{h^*}=0.4588$,	$\gamma_1=0.5770$,	$\gamma_2=0.5539$
	5b	$CC^*=0.6875$,	$\sqrt[4]{s^*}=0.6489$,	$\sqrt{f^*}=0.4472$,	$\sqrt[4]{h^*}=0.4729$,	$\gamma_1=0.5806$,	$\gamma_2=0.5609$
6) Fig.	6a = 6b	$CC^*=0.5556$,	$\sqrt[4]{s^*}=0.6455$,	$\sqrt{f^*}=0.4564$,	$\sqrt[4]{h^*}=0.4564$,	$\gamma_1=0.5739$,	$\gamma_2=0.5510$
9) Fig.	8	$CC^*=1.000$,	$\sqrt[4]{s^*}=1.000$,	$\sqrt{f^*}=1.000$,	$\sqrt[4]{h^*}=1.000$,	$\gamma_1=1.000$,	$\gamma_2=1.000$
10) Fig.	11a	$CC^*=0.8000$,	$\sqrt[4]{s^*}=0.5318$,	$\sqrt{f^*}=0.4472$,	$\sqrt[4]{h^*}=0.5318$,	$\gamma_1=0.5318$,	$\gamma_2=0.5318$
	11b	$CC^*=0.7333$,	$\sqrt[4]{s^*}=0.4969$,	$\sqrt{f^*}=0.3944$,	$\sqrt[4]{h^*}=0.3670$,	$\gamma_1=0.4460$,	$\gamma_2=0.4320$
	11c	$CC^*=0.7333$,	$\sqrt[4]{s^*}=0.5191$,	$\sqrt{f^*}=0.3944$,	$\sqrt[4]{h^*}=0.3670$,	$\gamma_1=0.4615$,	$\gamma_2=0.4431$
11) Fig.	12a	$CC^*=0.7037$,	$\sqrt[4]{s^*}=0.6467$,	$\sqrt{f^*}=0.4714$,	$\sqrt[4]{h^*}=0.5732$,	$\gamma_1=0.6132$,	$\gamma_2=0.6100$
	12b	$CC^*=0.7037$,	$\sqrt[4]{s^*}=0.6467$,	$\sqrt{f^*}=0.4714$,	$\sqrt[4]{h^*}=0.5973$,	$\gamma_1=0.6235$,	$\gamma_2=0.6220$
12) Fig.	13	$CC^*=0.8500$,	$\sqrt[4]{s^*}=0.8801$,	$\sqrt{f^*}=0.7746$,	$\sqrt[4]{h^*}=0.7347$,	$\gamma_1=0.8171$,	$\gamma_2=0.8074$

Proof. Since s^* and h^* both satisfy the principles (P_1) through (P_6), it is easy to see that γ^* also satisfies principles (P_1) through (P_6). Furthermore, it was shown that s^* is concave and h^* is strictly concave. Hence, γ^* is strictly concave. So (P_7) is also satisfied. Lastly, s^* satisfies (P_8), while h^* satisfies a weaker form of (P_8): let h_1^* resp. h_2^* be the values of h^* in the two situations described in (P_8). Then $h_1^* \leqq h_2^*$ (hence $<$ in (P_8) is only $\leqq$ here). The proof goes as follows (we use the same notation as in the proof that s^* satisfies (P_8)): In situation 1, (i_1, i_1') appeared already together in a box of the common situation. If this box is of size 2, then, denoting $D = 1/[n^2(n-1)^2Nk^2]$,

$$h_1^* = D\left(\sum_{\substack{i,i'=1 \\ i\neq i' \\ i,i'\neq i_1,i_1'}}^{n} \sum_{j=2}^{q} \sqrt{x_{i,i'}^{(j)}} + 2\sum_{j=3}^{q} \sqrt{x_{i_1,i_1'}^{(j)}} + 2\sqrt{x_{i_1,i_1'}^{(2)}} + 1\right). \tag{34}$$

If this box is not of size 2, then

$$h_1^* = D\left(\sum_{\substack{i,i'=1 \\ i\neq i' \\ i,i'\neq i_1,i_1'}}^{q} \sqrt{x_{i,i'}^{(j)}} + 2\sum_{j=2}^{q} \sqrt{x_{i_1,i_1'}^{(j)}} + 2\right). \tag{35}$$

In all cases, h_2^* is

$$h_2^* = D\left(\sum_{\substack{i,i'=1 \\ i\neq i'}}^{n} \sum_{j=2}^{q} \sqrt{x_{i,i'}^{(j)}} + 2\right). \tag{36}$$

Hence $h_2^* - h_1^*$ is, in case (34) is valid:

$$h_2^* - h_1^* = 2 + 2\sqrt{x_{i_1,i_1'}^{(2)}} - 2\sqrt{x_{i_1,i_1'}^{(2)}} + 1 > 0$$

since $x_{i_1,i_1'}^{(2)} > 0$ (since the box $\boxed{i_1, i_1'}$ existed already in the common situation). In case (35) is valid, we simply have that $h_1^* = h_2^*$ (cf. the example in Fig. 11), showing that, in general,

$$h_1^* \leqq h_2^*.$$

This weaker form, together with the fact that s^* satisfies (P_8), is enough to ensure that γ^* also satisfies (P_8). In conclusion, γ^* satisfies all principles (P_1) through (P_8). □

γ^* is easily calculated via the formula (cf. (23) and (31)):

$$\begin{aligned}\gamma^* &= \frac{1}{2n^2(n-1)^2N}\left[\left(\sum_{\substack{i,i'=1 \\ i\neq i'}}^{n} \sqrt{\sum_{j=2}^{q} x_{i,i'}^{(j)}}\right)^2 + \frac{1}{k}\left(\sum_{\substack{i,i'=1 \\ i\neq i'}}^{n} \sum_{j=2}^{q} \sqrt{x_{i,i'}^{(j)}}\right)^2\right] \\ &= \frac{1}{2n^2(n-1)^2N}\left[\left(\sum_{\substack{i,i'=1 \\ i\neq i'}}^{n} \sqrt{\sum_{j\in K} x_{i,i'}^{(j)}}\right)^2 + \frac{1}{k}\left(\sum_{\substack{i,i'=1 \\ i\neq i'}}^{n} \sum_{j\in K} \sqrt{x_{i,i'}^{(j)}}\right)^2\right].\end{aligned} \tag{37}$$

Although γ^* has all the good properties we have discussed, it is not our definitive measure of collaboration.

IV.6. *Polishing γ^**

Since we had to divide s by $n^2(n-1)^2N$ to get s^* and h by $n^2(n-1)^2Nk^2$ to get h^*, the actual values of s^* and h^* are very small. Certainly in practice, actual collaboration situations are far from "maximal" and hence, the values of s^* and h^* are skewed in the direction of 0. Examples of this are found at the end of this paper in Table 1. However, it is easy to remedy this, without giving up the properties (P_1) through (P_8). Good functions to use in combination with s^* and h^* are the power functions $x^{1/a} = \sqrt[a]{x}$ with $a > 1$. What a to use can be determined via the following "heuristic" property (it is certainly not a ninth principle): if the collaborative situation is 50% of what it could be, then $\sqrt[a]{h^*}$ and $\sqrt[a]{s^*}$ must be 0.5, approximately.

A collaboration of 50% could be expressed as in Fig. 14: Take $n \in \mathbf{N}$ even and N boxes each containing the objects $1,2,\ldots,(n/2) \in \mathbf{N}$. Since in this case all boxes have the same size, $s^* = h^* = \gamma^*$ (cf. the important note in section IV.4). In order to have that these functions, composed with $\sqrt[a]{\ }$, are 0.5 (approximately) in the case of Fig. 14, we must have that (use $\sqrt[a]{s^*}$ or $\sqrt[a]{h^*}$)

$$\left(\frac{1}{n^2(n-1)^2N}\left(\sqrt{N}\,\frac{n}{2}\left(\frac{n}{2}-1\right)\right)^2\right)^{1/a} \approx 0.5.$$

This yields

$$\left(\frac{(n-2)^2}{16(n-1)^2}\right)^{1/a} \approx 0.5.$$

Using that $(n-2)/(n-1) \approx 1$, we find

$$\frac{1}{16^{1/a}} \approx 0.5,$$

hence $a \approx 4$. We can now introduce two collaborative measures that satisfy all principles (P_1) through (P_8) together with the above heuristic property

$$\gamma_1 = \sqrt[4]{\gamma^*} \tag{38}$$

or

$$\gamma_2 = \frac{\sqrt[4]{s^*} + \sqrt[4]{h^*}}{2}. \tag{39}$$

It is clear that γ_1 as well as γ_2 satisfy all collaborative principles (P_1) through (P_8) (mainly since $\sqrt[4]{\ }$ is strictly increasing and concave) and that their values are reasonably spread around 0.5, which is good to have (in a heuristic way).

This last heuristic remark can also be made for f^*. Here one finds, for the situation in Fig. 14:

$$\sqrt[a]{f^*} = \sqrt[a]{\frac{Nn(n-2)}{4n(n-1)N}} \approx \sqrt[a]{\frac{1}{4}} = 0.5$$

for $a = 2$. Hence use $\sqrt{f^*}$ (if you want to use it; better not, since $\sqrt{f^*}$ lacks principle (P_7) as well as (P_8)). For CC^*, a cannot be found since, in the case of Fig. 14:

$$CC^* = \frac{n}{n-1}\left(1 - \frac{N\frac{2}{n}}{N}\right) \approx 1,$$

hence no reasonable a exists such that $\sqrt[a]{CC^*} = 0.5$. This reveals another drawback of CC^* (or CC): In the "50%" collaboration case, $CC \approx CC^* \approx 1$, hence a value close to the value of the "maximal" collaboration.

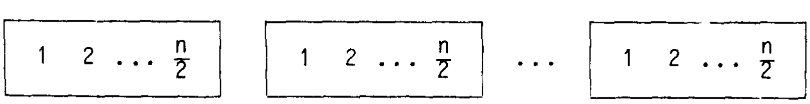

Fig. 14. "50%" collaboration.

IV.8. Relations between the collaborative measures

From the above it follows that the measures CC, f, s, and h are best represented by CC^*, $\sqrt{f^*}$, $\sqrt[4]{s^*}$, and $\sqrt[4]{h^*}$, while the measures γ_1 and γ_2 are the ultimate best collaborative measures (in this paper). In this section we will investigate some mathematical inequalities that exist between the measures $\sqrt[4]{s^*}$, $\sqrt[4]{h^*}$, γ_1, and γ_2.

PROPOSITION IV.8.1

$$\gamma_1 \geqq \gamma_2. \tag{40}$$

Proof. Using the Cauchy-Schwartz inequality in $\mathbf{R}^2$ on the vectors $(\mathbf{x},\mathbf{y})$ and $(\frac{1}{2},\frac{1}{2})$ one sees that, if $\mathbf{x},\mathbf{y} > 0$,

$$\frac{\mathbf{x}+\mathbf{y}}{2} \leqq \sqrt{\frac{\mathbf{x}^2+\mathbf{y}^2}{2}},$$

hence

$$\frac{\mathbf{x}+\mathbf{y}}{\sqrt{2}} \leqq \sqrt{\mathbf{x}^2+\mathbf{y}^2}.$$

Now this inequality will be applied twice: once for the vectors $\left(\sqrt[4]{\mathbf{x}},\sqrt[4]{\mathbf{y}}\right)$ and $(\frac{1}{2},\frac{1}{2})$, and then once for the vectors $\left(\sqrt{\mathbf{x}},\sqrt{\mathbf{y}}\right)$ and $(\frac{1}{2},\frac{1}{2})$ (each time for $\mathbf{x},\mathbf{y} > 0$). This gives:

$$\frac{\sqrt[4]{\mathbf{x}}+\sqrt[4]{\mathbf{y}}}{\sqrt{2}} \leq \sqrt{\sqrt{\mathbf{x}}+\sqrt{\mathbf{y}}}$$

$$= \sqrt[4]{2}\sqrt{\frac{\sqrt{\mathbf{x}}+\sqrt{\mathbf{y}}}{2}}$$

$$\leq \sqrt[4]{2}\sqrt[4]{\mathbf{x}+\mathbf{y}}.$$

Hence

$$\frac{\sqrt[4]{\mathbf{x}}+\sqrt[4]{\mathbf{y}}}{\sqrt{2}} \leqq \sqrt{2}\sqrt[4]{\frac{\mathbf{x}+\mathbf{y}}{2}}.$$

So, for every $\mathbf{x},\mathbf{y} > 0$,

$$\sqrt[4]{\frac{\mathbf{x}+\mathbf{y}}{2}} \geqq \frac{\sqrt[4]{\mathbf{x}}+\sqrt[4]{\mathbf{y}}}{2}.$$

Interpreted for $x = s^*$ and $y = h^*$ this is

$$\gamma_1 \geqq \gamma_2.$$ □

PROPOSITION IV.8.2

$$h^* \leqq s^*. \tag{41}$$

Proof. Using the Cauchy-Schwarz inequality with second vector $(1,1,\ldots,1) \in \mathbf{R}^k$ we find, for every $i \neq i'$:

$$\sum_{j\in K} \sqrt{\mathbf{x}_{i,i'}^{(j)}} \leqq k\sqrt{\sum_{j\in K} \mathbf{x}_{i,i'}^{(j)}}.$$

Hence

$$\sum_{i,i',j'} \sqrt{\mathbf{x}_{i,i'}^{(j)}} \leqq k \sum_{\substack{i,i'=1\\ i\neq i'}}^{n} \sqrt{\sum_{j\in K} \mathbf{x}_{i,i'}^{(j)}}.$$

Consequently

$$\frac{1}{n^2(n-1)^2Nk^2}\left(\sum_{i,i',j} \sqrt{\mathbf{x}_{i,i'}^{(j)}}\right)^2 \leqq \frac{1}{n^2(n-1)^2N}\left(\sum_{\substack{i,i'=1\\ i\neq i'}}^{n} \sqrt{\sum_{j\in K} \mathbf{x}_{i,i'}^{(j)}}\right)^2,$$

which is

$$h^* \leqq s^*. \qquad \square$$

COROLLARY IV.8.3

We have the following inequalities:

$$\sqrt[4]{h^*} \leqq \gamma_2 \leqq \gamma_1 \leqq \sqrt[4]{s^*}. \tag{42}$$

Proof. Both γ_1 and γ_2 are intermediate between the values $\sqrt[4]{h^*}$ and $\sqrt[4]{s^*}$ by their definition (formulae (37), (38) and (39)). Furthermore, (41) implies

$$\sqrt[4]{h^*} \leqq \sqrt[4]{s^*},$$

hence (42) is proved. $\square$

It is not clear what the relation between CC^*, $\sqrt{f^*}$ and the better measures above are.

IV.9. *Examples*

To see better how the above measures CC^*, $\sqrt[4]{s^*}$, $\sqrt{f^*}$, $\sqrt[4]{h^*}$, γ_1, and γ_2 act in practical examples, we have calculated their values in some examples of collaborative situations we came across so far (referring to the figure number) as well as in new examples, some of which are collected in Table 1. One should notice that in many cases of comparison, the measures CC^* and $\sqrt{f^*}$ usually give no different values, while $\sqrt[4]{s^*}$, $\sqrt[4]{h^*}$, γ_1, and γ_2 do.

The reader can verify the discussed principles and see that the measures γ_1 and γ_2 are ultimately best. Examples in addition to those in Table 1 follow.

1. We repeat that in case of Fig. 14 we have

$$CC^* = \frac{n-2}{n-1} \approx 1, \quad \sqrt[4]{s^*} = \sqrt[4]{h^*} = \frac{1}{2}\sqrt{\frac{n-2}{n-1}} \approx 0.5, \quad \sqrt{f^*} = \frac{1}{2}\sqrt{\frac{n-2}{n-1}} \approx 0.5,$$

$$\gamma_1 \approx 0.5, \quad \gamma_2 \approx 0.5.$$

2. Consider the situations with respect to Fig. 15: (a) $n = 2$, (b) $n = 3$ (objects: 1,2,3). Then we have

(a) $CC^* = 1.000$, $\sqrt[4]{s^*} = 1.000$, $\sqrt{f^*} = 1.000$, $\sqrt[4]{h^*} = 1.000$, $\gamma_1 = 1.000$, $\gamma_2 = 1.000$.
(b) $CC^* = 0.7500$, $\sqrt[4]{s^*} = 0.5774$, $\sqrt{f^*} = 0.5774$, $\sqrt[4]{h^*} = 0.5774$, $\gamma_1 = 0.5774$, $\gamma_2 = 0.5774$.

3. Consider the situations ($n = 3$) with respect to Fig. 16. So, in comparison, to form the collaboration in Fig. 16b, one collaboration $\boxed{1\ 2}$ of Fig. 16a has been disconnected, but one collaboration $\boxed{2\ 3}$ has been added. We have the following values. For Fig. 16a:

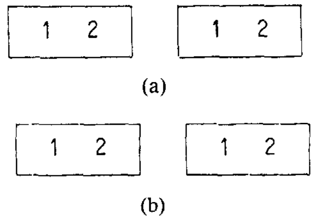

Fig. 15. a. $n = 2$. b. $n = 3$.

$$\sqrt[4]{h^*} = \sqrt[4]{s^*} = \gamma_1 = \gamma_2$$
$$= \left(\frac{1}{3^2 2^2(\alpha + \beta)} (2\sqrt{\alpha} + 2\sqrt{\beta})^2\right)^{1/4} \text{ (call this } v_1).$$

For Fig. 16b:

$$\sqrt[4]{h^*} = \sqrt[4]{s^*} = \gamma_1 = \gamma_2$$
$$= \left(\frac{1}{3^2 2^2(\alpha + \beta)} (2\sqrt{\alpha - 1} + 2\sqrt{\beta + 1})^2\right)^{1/4} \text{ (call this } v_2).$$

Then $v_1 < v_2$ if and only if $\sqrt{\alpha} + \sqrt{\beta} < \sqrt{\alpha - 1} + \sqrt{\beta + 1}$ if and only if $\sqrt{\alpha} - \sqrt{\alpha - 1} < \sqrt{\beta + 1} - \sqrt{\beta}$ if and only if $\alpha > \beta + 1$. Since we deal here with $\alpha, \beta \in \mathbf{N}$, this inequality is equivalent to

$$1 + \beta \leqq \alpha - 1. \tag{43}$$

This condition expresses that we can "add" more collaborations $\boxed{2\ 3}$ as long as we do not end up with more collaborations $\boxed{2\ 3}$ than collaborations $\boxed{1\ 2}$. So our measures are so sensible that they more highly value the cases with more equal balance of collaboration. This is a very good property and is a consequence of the concavity of these measures (note that, since here $\sqrt[4]{s^*} = \sqrt[4]{h^*}$, this concavity is also valid for $\sqrt[4]{s^*}$, but only because we only have boxes with the same number of objects (here, 2)). The measures CC^* and $\sqrt{f^*}$ lack this property: In both cases (Fig. 16a and b) we have $CC^* = 0.75$ and $\sqrt{f^*} = 0.5774$, hence even independent of α/β (which would be the least to be expected!).

4. Consider the situations presented in Fig. 17. Here we have, in the case of Fig. 17a:

$$CC^* = 0.6222, \quad \sqrt[4]{s^*} = 0.3523, \quad \sqrt{f^*} = 0.2582, \quad \sqrt[4]{h^*} = 0.2491,$$
$$\gamma_1 = 0.3132, \quad \gamma_2 = 0.3007.$$

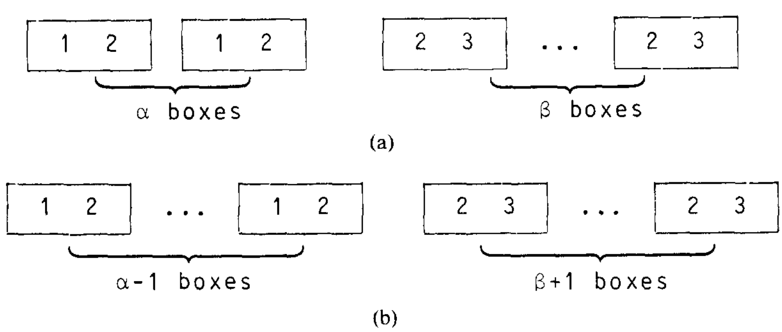

Fig. 16. $n = 3$.

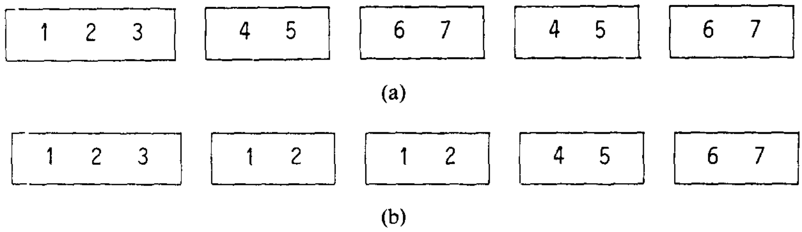

Fig. 17.

In the case of Fig. 17b:

$$CC^* = 0.6222, \quad \sqrt[4]{s^*} = 0.3494, \quad \sqrt{f^*} = 0.2582, \quad \sqrt[4]{h^*} = 0.2613,$$

$$\gamma_1 = 0.3145, \quad \gamma_2 = 0.3054.$$

5. Consider the situation presented in Fig. 18 (for $n = 8$, but only the objects 1,2,3,4 collaborate). Then:

$$CC^* = 0.8571, \quad \sqrt[4]{s^*} = 0.4629, \quad \sqrt{f^*} = 0.4629, \quad \sqrt[4]{h^*} = 0.4629,$$

$$\gamma_1 = 0.4629, \quad \gamma_2 = 0.4629.$$

We again see that CC^* does not measure the heuristic "norming" principle of "50%" collaboration very well. The same conclusion can be made in Fig. 19: n = 20. Here

$$CC^* = 0.9474, \quad \sqrt[4]{s^*} = 0.4867, \quad \sqrt{f^*} = 0.4867, \quad \sqrt[4]{h^*} = 0.4867,$$

$$\gamma_1 = 0.4867, \quad \gamma_2 = 0.4867.$$

In fact, the higher n, $CC^* \to 1$ while $\sqrt[4]{s^*}, \sqrt{f^*}, \sqrt[4]{h^*}, \gamma_1, \gamma_2 \to 0.5$ (for $N = n/2$) in the above examples.

6. We close with an example, in Fig. 20, of two unequal situations (even unequal up to a permutation) that have the same values of the measures studied so far.

In *both* situations we have the following set of measures:

$$CC^* = 0.6806, \quad \sqrt[4]{s^*} = 0.4690, \quad \sqrt{f^*} = 0.3563, \quad \sqrt[4]{h^*} = 0.3316,$$

$$\gamma_1 = 0.4170, \quad \gamma_2 = 0.4003.$$

The attentive reader might note that we had this already with the examples 7a versus 11b and 7b versus 11c. But in both these cases, the two situations are identical, up to a permutation (as is readily seen). In the above situation, however, such a permutation does not exist. Indeed, if a permutation of {1,2,3,4,5,6,7} existed such that Fig. 20a is transformed in Fig. 20b, then necessarily (look at the boxes with four elements) 1 is the image of 4, 5, 6, or 7. This yields a contradiction when looking at the boxes with two elements.

This is not a bad property, but just a remark: one cannot expect our measures γ_1, γ_2 to make distinction between any two different situations; γ_1 and γ_2 only make distinction

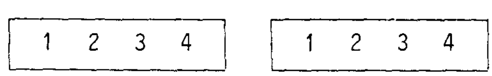

Fig. 18. $n = 8$, but only objects 1, 2, 3, 4 collaborate.

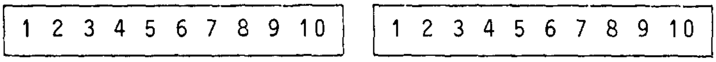

Fig. 19. $n = 20$.

for the cases described in the principles (P_i). Besides, in the above example, it is not clear whether we are dealing here with two different situations (from the point of view of collaboration).

V. SUMMARY

In this paper we discussed collaborative measures. In order to be able to do this, one must first define the variables out of which such measures can be constructed.

We showed that the variables $(f_j)_{j=1,\ldots,q}$ and $(x_{i,i'})_{i,i'=1,\ldots,n}$, found in separate studies in the past ([1],[5]) are not even sufficient *together*. Therefore we used the finer variables

$x_{i,i'}^{(j)}$ = the number of times objects i and i' are together in a box that contains j objects $(i,i' = 1,\ldots,n; j = 1,\ldots,q)$.

Given these variables, we next need to know what are "good" properties of collaboration. We distinguished eight principles, which are natural.

Then two known measures (CC,s) as well as two new measures (f,h) were studied. Normalization was needed in all cases in order to have values less than one. These measures were denoted by CC^*, s^*, f^*, and h^*.

We found that CC^* and f^* satisfy the first six principles and that s^* and h^* each satisfy seven principles: the first six principles; for s^*, the Bridging Principle; and for h^*, the strict concavity principle.

We were then able to construct two measures γ_1, γ_2 that satisfy all eight principles:

$$\gamma_1 = \sqrt[4]{\frac{s^* + h^*}{2}} \tag{44}$$

$$\gamma_2 = \frac{\sqrt[4]{s^*} + \sqrt[4]{h^*}}{2} \tag{45}$$

where

$$s^* = \frac{1}{n^2(n-1)^2N}\left(\sum_{\substack{i,i'=1\\ i\neq i'}}^{n}\sqrt{\sum_{j\in K}^{k} x_{i,i'}^{(j)}}\right)^2 \tag{26}$$

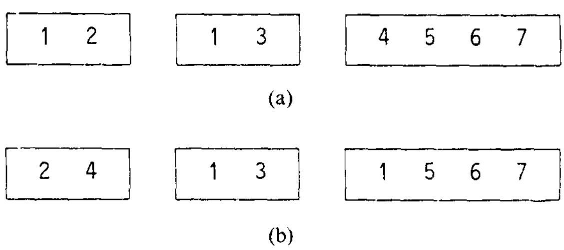

Fig. 20. Two unequal situations.

Table 2. Properties to the measures studied (Y = yes, N = no)

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	H
CC^*	Y	Y	Y	Y	Y	Y	N	N	N
$\sqrt[4]{s^*}$	Y	Y	Y	Y	Y	Y	N	Y	Y
$\sqrt{f^*}$	Y	Y	Y	Y	Y	Y	N	N	Y
$\sqrt[4]{h^*}$	Y	Y	Y	Y	Y	Y	Y	N	Y
γ_1	Y	Y	Y	Y	Y	Y	Y	Y	Y
γ_2	Y	Y	Y	Y	Y	Y	Y	Y	Y

and

$$h^* = \frac{1}{n^2(n-1)^2Nk^2}\left(\sum_{i,i',j}\sqrt{x_{i,i'}^{(j)}}\right)^2 \tag{34}$$

In addition, we could show that $\sqrt[4]{s^*}$, $\sqrt{f^*}$, $\sqrt[4]{h^*}$, γ_1, and γ_2 also satisfy the "heuristic" property (H) that one should find a value 0.5 (approximately) if we have "50%" collaboration. CC^* is not adaptable in this way. This "heuristic" property serves our purpose of comparing the different measures.

Table 2 relates the properties to the measures studied (Y = yes, N = no).

The following inequalities could be proved:

$$\sqrt[4]{h^*} \leqq \gamma_2 \leqq \gamma_1 \leqq \sqrt[4]{s^*}.$$

An extensive number of examples was given, illustrating the properties proved earlier.

It is our hope that, in studies of collaboration (e.g., science policy studies, social studies, . . .) one of the measures γ_1, γ_2 will be preferred above the other ones; their calculation is very simple and does not require human interpretation.

It would be interesting to see whether other collaborative properties (principles) are needed and, together with this, if other good (better) collaborative measures can be constructed.

Finally, we end with a few open problems:

1. Refine principle (P_7) ("strict concavity") so that it becomes a "perfect" property for collaborative measures on the variables $x_{i,i'}^{(j)}$.
2. A thorough study of the "Bridging Principle" (P_8) and extensions thereof must still be done.
3. Develop collaborative theories on other variables than the variables $x_{i,i'}^{(j)}$.
4. As remarked to me by Prof. Englisch, a collaborative situation is nothing but a "hypergraph" (cf. [8]). Therefore, it is an open problem to study collaborative measures in the connection of hypergraphs. In fact, it is our opinion that hypergraphs must have their applications in more aspects of informetrics (such as citation analysis).
5. Make a collaborative theory involving substructures (subgroups), as appears in the experimental work of H. Kretschmer [9] (and many other older references to Kretschmer's work).

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