Pergamon

Information Processing & Management, Vol. 31, No. 4, pp. 511–523, 1995

0306-4573/95 $9.50+0.00

0306-4573(95)00005-4

SENSITIVITY ASPECTS OF INEQUALITY MEASURES

LEO EGGHE[1,3] and RONALD ROUSSEAU[2,3]

[1] LUC, Universitaire Campus, B-3590, Diepenbeek, Belgium
[2] KIHWV, Zeedijk 101, B-8400, Oostende, Belgium
[3] UIA, Informatie- en Bibliotheekwetenschap, Universiteitsplein 1, B-2610, Wilrijk, Belgium

(Received August 1994; accepted in final form January 1995)

Abstract—The purpose of this article is to study inequality measures with respect to their sensitivity to transfers. Sensitivity is studied by means of a particular directional derivative. We observe that inequality measures behave differently in the sense that the vectors for which this directional derivative is positive or negative differ according to the used inequality measure. It is shown that different averages, such as the arithmetic mean, the median, the harmonic mean and the geometric mean play an essential role in these investigations. We conclude that the use of this directional derivative introduces a battery of sensitivities in the class of inequality measures. This helps the information scientist to choose between otherwise acceptable measures.

Keywords: Inequality measures, Sensitivity to transfers, Averages.

1. INTRODUCTION

Inequality measures (measures of concentration or diversity) play an important role in several scientific domains such as the information sciences (Burrell, 1991; Rousseau, 1993), economics (Atkinson, 1970; Sen, 1973; Rousseau, 1992), sociology (Allison, 1978), demography (White, 1986), linguistics (Herdan, 1966) and ecology (Magurran, 1991; Rousseau & Van Hecke, 1993). The field of information science is especially characterized by a high degree of inequality (Egghe, 1987).

The purpose of this article is to study inequality measures with respect to their sensitivity to transfers. Sensitivity is studied by means of a particular directional derivative. We observe that inequality measures behave differently in the sense that the vectors for which this directional derivative is positive or negative differ according to the used concentration or diversity measure. It is shown that different averages, such as the arithmetic mean, the median, the harmonic mean and the geometric mean play an essential role in these investigations. We conclude that the use of this directional derivative introduces a battery of sensitivities in the class of inequality measures. This helps the information scientist to choose between otherwise acceptable measures. Before studying these sensitivity aspects, we will recall how concentration measures are defined.

When studying inequality of discrete situations the terminology of cells and balls, to be deposited into cells, is often used; the words sources and items are also common. In practical situations cells and balls will be researchers and the number of articles they publish, or books and the number of loans during a fixed period, and so on. If the number of cells is N, we denote by x_i, $i=1, \ldots, N$, the number of balls in the ith cell. The assumption is that all x_i are non-negative and that at least one x_i is different from zero.

To every distribution of items over sources, this is to every N-vector $X=(x_1, \ldots, x_N)$ an inequality measure associates a positive number characterizing the inequality (concentration or diversity) in this distribution. To qualify as an inequality measure a function from $\mathbf{R}^N$ to $\mathbf{R}^+$ must satisfy the following axioms.

(P) Permutation invariance (anonymity rule)

Let f be a function: $\mathbf{R}^N \to \mathbf{R}^+$ and let π be a permutation of the set $\{1, \ldots, N\}$, then this axiom states that for every vector $(x_1, x_2, \ldots, x_N)$ in $\mathbf{R}^N$

$$f(x_1, x_2, \ldots, x_N) = f(x_{\pi(1)}, \ldots, x_{\pi(N)}). \tag{1}$$

Equality (1) means that concentration and diversity are not properties of individuals but of a group as a whole.

(S) Scale invariance

The function value characterizing concentration or diversity should not depend on the used units. Stated as an equality this axiom is: if f is a function from $\mathbf{R}^N$ to $\mathbf{R}^+$ then f is said to be scale invariant if for every vector $X = (x_1, x_2, \ldots, x_N)$ and every $c > 0$:

$$f(cx_1, cx_2, \ldots, cx_N) = f(x_1, x_2, \ldots, x_N). \tag{2}$$

As we consider situations described in (P) and (S) as indistinguishable, we will say that two N-tuples are equivalent when they differ in the order of their coordinates, or when one is a positive multiple of the other, or one vector can be derived from the other by a permutation of the coordinates and a multiplication by a strictly positive constant. When vectors X and X' are equivalent, this is denoted as $X \equiv X'$. As the value of a function which satisfies (P) and (S) is the same for every vector of an equivalence class we will use the same notation for all and simply write $f(X) = f(x_1, x_2, \ldots, x_N)$ for the value which f takes in any vector $X = (x_1, x_2, \ldots, x_N)$, of this class.

(T) The transfer principle (concentration form)

This principle, proposed by Dalton (1920), states that a strictly positive transfer from a poorer source to a richer one, must lead to a strict increase in concentration. Put in a mathematical framework (T) becomes: for every N-vector $(x_1, x_2, \ldots, x_N)$, $x_i < x_j$ and $0 < h < x_i$, we have:

$$f(x_1, x_2, \ldots, x_i - h, \ldots, x_j + h, \ldots, x_N) > f(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots, x_N). \tag{3}$$

In our opinion, the set $\{(P), (S), (T)\}$ forms a minimum set of requirements a concentration measure must satisfy. Thus we will use the following definition.

DEFINITION 1: *A concentration measure*

A function $f: \mathbf{R}^N \to \mathbf{R}^+$ is termed a type I concentration measure if f satisfies the axioms (P), (S) and (T).

Note. The addition "of type I" refers to the fact that we have introduced [see Egghe & Rousseau (1991)] measures of type II and type III. These measures, however, will play no role in this article.

DEFINITION 2: *The Lorenz dominance order*

Let $X = (x_1, \ldots, x_N)$ and $X' = (x'_1, \ldots, x'_N)$ be N-vectors. The vector X' is said to majorize the vector X in the Lorenz dominance order (Hardy *et al.*, 1952, p. 45) when the following three requirements are satisfied:

(i) $x_1 \leq x_2 \leq \cdots \leq x_N;\ x'_1 \leq x'_2 \leq \cdots \leq x'_N;$ (4)

(ii) for every $i = 1, \ldots, N$ we have:

$$\sum_{j=1}^{i} x_j \geq \sum_{j=1}^{i} x'_j \tag{5}$$

$$\text{(iii)} \sum_{j=1}^{N} x_j = \sum_{j=1}^{N} x_j'. \tag{6}$$

The Lorenz dominance order compares vectors with a different concentration. When X is majorized by X' this is denoted as $X \prec X'$. The relation $\prec$ determines a partial order in the set of all N-vectors. The partial order $\prec$ restricted to a fixed N has a smallest element, namely $\mathbf{0}=(x, x, x, \ldots, x)$, $x \neq 0$. This vector denotes the equality situation. The monopoly situation $\mathbf{1}=(0, \ldots, 0, y)$, $y>0$, represents the largest element for $\prec$. When N-vectors are comparable with respect to the Lorenz dominance order, we will say that they are intrinsically comparable.

When $X \prec X'$ this means that the Lorenz curve of X' lies nowhere above the Lorenz curve of X. Of course, these curves may coincide over some region. [For a description of the Lorenz curve and its construction the reader is referred to Rousseau (1993).]

Every inequality measure imposes on the set of all N-vectors an order which is strictly finer than the Lorenz dominance order. Indeed, crossing Lorenz curves will usually have different values for a given inequality measure. Moreover, different measures often rank crossing Lorenz curves differently. Stated otherwise, the choice of an inequality measures involves a decision regarding the concentration or diversity order of situations that are not intrinsically comparable. To make this decision, one often considers the sensitivity of inequality functions with respect to an elementary transfer (Atkinson, 1970). In this article we will study another sensitivity aspect, related to more realistic transfers, with the aim to have more information by which to choose between different, otherwise acceptable, inequality functions.

In the following section we will moreover assume that all concentration measures we will study are differentiable in every direction.

2. A DIFFERENT KIND OF TRANSFER

In Egghe and Rousseau (1990) we made the following simple observation.

PROPOSITION 1 *(Egghe & Rousseau, 1990; Proposition 5.5)*

Let f be a concentration measure and let x_N be the largest of all the components x_i. Then:

if $0<h<\min_i (N-1)x_i$ the following inequality holds:

$f(X')>f(X)$, where $X=(x_1, \ldots, x_N)$ and

$$X'=\left(x_1-\frac{h}{N-1}, x_2-\frac{h}{N-1}, \ldots, x_{N-1}-\frac{h}{N-1}, x_N+h\right). \tag{7}$$

This inequality is an immediate consequence of the transfer principle. Now, one may rightly observe that the requirements of this proposition are rather artificial. What happens if the second richest person becomes richer at the cost of all the others? Or another person? Note that this is a realistic question. Moonlighting, for instance, leads to the enrichment of the moonlighter but is, by not paying taxes, detremental to all other citizens. For this reason, we will consider more general transfers.

Before proceeding we note that, because of the anonymity rule, we may assume without loss of generality that the main transfer, i.e. $+h$, occurs at the first component. Whenever possible we will make this assumption. Now, putting $x=(x_1, \ldots, x_N)$ and $X'=(x_1+h, x_2-h/(N-1), \ldots, x_N-h/(N-1))$, with $0<h<\min_i(N-1)x_i$ it seems plausible to require that for a good concentration measure f, $f(X')>f(X)$, i.e. concentration increases, if the person who increases his income, i.e. x_1, had already an income above average. Therefore we could define a new transfer principle for concentration measures as follows: if $x_1>\mu(X)$ then $f(X')>f(X)$, with X' as above. Similar definitions can be formulated with $\mu(X)$ replaced by HM(X), GM(X), Md(X) (the harmonic mean, respectively the geometric mean, respectively the median of X). We will, however, go one step further.

3. A NEW TRANSFER PRINCIPLE AND A MEASURE OF SENSITIVITY FOR INEQUALITY MEASURES

The transfer principle defined above, as well as the "classical" transfer principle as defined in Section 1, only compare two situations: X (before the transfer) and X' (after the transfer). In these cases a total transfer of $h>0$ is involved. Inequalities change if $h<0$ and any "small" $|h|$ can be used. This boils down to calculating (for every "small" h)

$$\frac{f(X')-f(X)}{h} \tag{8}$$

and examining the sign of this number. Instead of using slopes of cords we could as well use the slope of a tangent in X, i.e. a directional derivative. This approach is similar to the one followed in (Egghe, 1994), where a theory of "continuous rates" has been presented.

Definition 3: *A directional derivative*

Let Y denote the following N-vector:

$$Y=\left(1, \frac{1}{N-1}, \ldots, -\frac{1}{N-1}, -\frac{1}{N-1}, \ldots, -\frac{1}{N-1}\right). \tag{9}$$

Then we consider the directional derivative of a measure f at the point X in the direction Y [cf. Protter & Morrey (1977), 16.6];

$$f'(X;Y)=\lim_{h\to 0}\frac{f(X+hY)-f(X)}{h},$$

$$=\sum_{i=1}^{N}\frac{\partial f}{\partial x_i}(X)y_i \tag{10}$$

$$=\frac{\partial f}{\partial x_1}(X)-\frac{1}{N-1}\sum_{i=2}^{N}\frac{\partial f}{\partial x_i}(X). \tag{11}$$

Putting $g_x : \mathbf{R}\to\mathbf{R} : h\to f(X+hY)$, we observe that $f'(X;Y)=g_x'(0)$.

Definition 4: *A new transfer principle*

Let f be a concentration measure. Then we say that f satisfies the transfer principle (T_M) if $x_1>M(X)$ implies

$$f'(X;Y)>0,$$

where $M(X)$ represents any "reasonable" average (such as the arithmetic, geometric or harmonic mean, or even the median).

Note that, we will only consider N-vectors with a fixed arithmetical average, equal to μ. Indeed, $\mu(X)=\mu(X+hY)$, for every h. The following two propositions show that the above definition leads to a meaningful transfer principle.

Proposition 2

Let $Y=(1, -1/(N-1), \ldots, -1/(N-1), -1/(N-1), \ldots, -1/(N-1))$, and assume that $\forall k$, $0\leq k<\min_{i\geq 2}(x_i)$: $f'(X+kY;Y)>0$, then for all h, $0<h<\min_{i\geq 2}(x_i)$

$$f\left(x_1+h, x_2-\frac{h}{N-1}, \ldots, x_N-\frac{h}{N-1}\right)>f(x_1, \ldots, x_N).$$

or $g_x(h)>g_x(0)$.

Proof. As g_x is differentiable (because f is), there exists a point k, $k \in]0, h[$ (by the mean value theorem) such that

$$g_x(h)-g_x(0)=g_x'(k)h.$$

Now, $g_x'(k)>0$ if and only if

$$\lim_{t\to 0}\frac{g_x(k+t)-g_x(k)}{t}>0$$

$$\Leftrightarrow$$

$$\lim_{t\to 0}\frac{f(X+(k+t)Y)-f(X+kY)}{t}>0$$

$$\Leftrightarrow \quad f'(X+kY; Y)>0,$$

which is true by assumption. This shows that $g_x(h)>g_x(0)$ or

$$f\left(x_1+h, x_2-\frac{h}{N-1}, \ldots, x_N-\frac{h}{N-1}\right)>f(x_1, \ldots, x_N).$$

Similarly, we state without proof.

Proposition 3

Let

$$Y=\left(1, -\frac{1}{N-1}, \ldots, -\frac{1}{N-1}, -\frac{1}{N-1}, \ldots, -\frac{1}{N-1}\right),$$

and assume that

$$\forall k: 0 \leq k < \min_{i\geq 2}(x_i): f'(X+kY; Y)<0,$$

then for all

$$h, 0<h\leq \min_{i\geq 2}(x_i)$$

$$f\left(x_1+h, x_2-\frac{h}{N-1}, \ldots, x_N-\frac{h}{N-1}\right)<f(x_i, \ldots, x_N),$$

or $g_x(h)<g_x(0)$.

In the applications we will show that, for example, for the variation coefficient V: if $x_1>\mu(X)$ then $V'(X; Y)>0$. Then, clearly, for $k>0$, $x_1+k>x_1>\mu(X)=\mu(X+kY)$, hence also $V'(X+kY; Y)>0$, so that the above proposition is applicable.

As measures of inequality are ordinal, any monotonic transformation of such a measure will preserve the ranking. If now ϕ is such a monotonic transformation and f_1 is type I measure then the directional derivatives of the functions f_1 and $f_2=\phi \circ f_1$ are related through the following proposition.

Proposition 4 *(Protter & Morrey, 1977, 16.8)*

If $\phi: \mathbf{R}\to\mathbf{R}$ is a differentiable function and f is a differentiable scalar field then $\forall X \in \mathbf{R}^N$:

$$(\phi \circ f)'(X; Y)=\phi'(f(X))f'(X; Y).$$

Proposition 4 allows us to restrict our investigations to one function of a family of equivalent ones (i.e. functions which can be derived from each other by a monotonic transformation).

The use of directional derivatives instead of values of $f(X')$, for fixed h, has another important consequence. Not only the sign of $f(X; Y)$ can be used, but also the value itself. This value represents the sensitivity of f with respect to the transfer $X \to X'$, $X' = X + hY$, with Y as in (9). Therefore we define:

DEFINITION 5: *a measure of sensitivity*

For any inequality measure f, the value $f'(X; Y)$ is defined as the sensitivity of f in X with respect to the transfer $X \to X + hY$, with Y as in (9).

Note: This approach could be compared with the use of Pearson's correlation coefficient, r, in statistics: the sign of r determines the kind of relation between two variables and r itself measures the degree of correlation.

4. THE DIRECTIONAL DERIVATIVE OF SOME WELL KNOWN INEQUALITY MEASURES

In this section we will study a number of well known concentration measures with respect to their sensitivity to transfers. We will find remarkable necessary and sufficient conditions for $f'(X; Y)$ to be positive or negative. Recall that in all our calculations μ is treated as a constant.

(A) The coefficient of variation (V)

$$V = \frac{\sigma}{\mu} \tag{12}$$

where σ denotes the standard deviation and μ denotes the arithmetic mean.

As

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2,$$

we find, by eqn (11)

$$V'(X; Y) = \frac{1/\mu\sqrt{N}}{\sqrt{\sum_{i=1}^{N} (x_i - \mu)^2}} \left((x_1 - \mu) - \sum_{i=2}^{N} \frac{(x_i - \mu)}{N-1} \right). \tag{13}$$

Hence $V'(X; Y) > 0$ if and only if $x_1 > \mu$. This is: this directional derivative is positive if and only if x_1 is larger than the arithmetic mean.

(B) Theil's first entropy measure (Th) (Theil, 1967)

$$\mathrm{Th} = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{x_i}{\mu}\right) \ln\left(\frac{x_i}{\mu}\right). \tag{14}$$

Then

$$\mathrm{Th}'(X; Y) = \frac{1}{N\mu} \left(\left(\ln\left(\frac{x_1}{\mu}\right) + \mu \right) - \frac{1}{N-1} \sum_{i=2}^{N} \left(\ln\left(\frac{x_i}{\mu}\right) + \mu \right) \right) \tag{15}$$

Now,

$$\mathrm{Th}'(X;Y)>0$$

$$\Leftrightarrow \quad \ln\left(\frac{x_1}{\mu}\right)+\mu>\frac{1}{N-1}\sum_{i=2}^{N}\left(\ln\left(\frac{x_i}{\mu}\right)+\mu\right)$$

$$\Leftrightarrow \quad \ln\left(\frac{x_1}{\mu}\right)>\frac{1}{N}\sum_{i=1}^{N}\ln\left(\frac{x_i}{\mu}\right)$$

$$\Leftrightarrow \quad \ln(x_1)>\frac{1}{N}\sum_{i=1}^{N}\ln(x_i)$$

$$\Leftrightarrow \quad x_1>\prod_{i=1}^{N}x_i^{1/N}.$$

This is: to have a strictly positive directional derivative, it is necessary and sufficient that x_1 is larger than the geometric mean of X.

(C) Theil's second entropy measure (L) (Theil, 1967)

$$L=-\left(\ln(N)+\frac{1}{N}\sum_{i=1}^{N}\ln(a_i)\right) \tag{16}$$

where $a_i=x_i/\mu N$.
Now,

$$L'(X;Y)=-\frac{1}{N}\left(\frac{1}{x_1}-\frac{1}{N-1}\sum_{i=2}^{N}\frac{1}{x_i}\right).$$

Hence:

$$L'(X;Y)>0$$

$$\Leftrightarrow \quad -\frac{1}{N}\left(\frac{1}{x_1}-\frac{1}{N-1}\sum_{i=2}^{N}\frac{1}{x_i}\right)>0$$

$$\Leftrightarrow \quad \frac{1}{x_1}<\frac{1}{N}\sum_{i=1}^{N}\frac{1}{x_i}$$

$$\Leftrightarrow \quad x_1>\frac{N}{\sum_{i=1}^{N}\frac{1}{x_i}}=\mathrm{HM}(X).$$

This shows that for Theil's second entropy measure, the directional derivative is strictly positive if and only if x_1 is larger than the harmonic mean of X, denoted as HM(X).

(D) The Gini index (G)

Assume that the components of the vector X are placed in increasing order. Then G is defined as:

$$G=\frac{N+1}{N}-\frac{2}{N}\sum_{i=1}^{N}(N+1-i)a_i. \tag{17}$$

Here we will assume that the main transfer occurs at the ith component. Then $G'(X;Y)$

$$=-\frac{2(N+1-i)}{\mu N^2}+\frac{2}{\mu N^2(N-1)}\sum_{j\neq i}(N+1-j).$$

Hence: $G'(X;Y)>0$

$$\Leftrightarrow \quad i>\frac{1}{N}\frac{N(N+1)}{2}$$

$$\Leftrightarrow \quad x_i > \text{median of } X.$$

The influence of this kind of transfer on the Gini index turns out to depend on the rank of the source where the main transfer occurs, and not on its actual value.

(E) The generalized Gini index: $G(r)$, $r \in N_0$ (Allison, 1978)

$$G(r)=\frac{\left(\frac{1}{2N(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{N}|x_i-x_j|^r\right)^{1/r}}{\mu}. \tag{18}$$

Then,

$$G(r)'(X;Y)=\left(\frac{\frac{1}{r}\left(\frac{1}{2N(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{N}|x_i-x_j|^r\right)^{(1-r)/r}}{\mu}\right).$$

$$\left(\frac{1}{2N(N-1)}\left(\sum_{j=1}^{N}\frac{r}{2}((x_1-x_j)^2)^{(r-2)/2}\,4(x_i-x_j)\right)\right.$$

$$\left.-\frac{1}{2N(N-1)^2}\left(\sum_{i=2}^{N}\sum_{j=2}^{N}\frac{4r}{2}(x_i-x_j)((x_i-x_j)^2)^{(r-2)/2}\right)\right).$$

Now, the second term of the second factor is zero, because the summation ranges over every i and every j, so with every term containing (x_i-x_j) there is also a term containing (x_j-x_i). Hence

$$G(r)'(X;Y)=\left(\frac{\frac{1}{r}\left(\frac{1}{2N(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{N}|x_i-x_j|^r)^{(1-r)/r}\right.}{\mu}\right).$$

$$\left(\frac{1}{2N(N-1)}\left(\sum_{j=1}^{N}\frac{r}{2}((x_i-x_j)^2)^{(r-2)/2}\,4(x_i-x_j\right)\right).$$

Consequently: $\quad G(r)'(X;Y)>0$

$$\Leftrightarrow \quad \sum_{j=1}^{N}(x_i-x_j)((x_i-x_j)^2)^{(r-2)/2}>0$$

$$\Leftrightarrow \quad \sum_{x_l>x_j}(x_1-x_j)^{r-1}>\sum_{x_1<x_j}(x_j-x_1)^{r-1}.$$

Here a special kind of average is involved. Note that for $r=1$ we once more find Gini's index. This is left to the reader.

(F) Atkinson's indices: A(e) (Atkinson, 1970)

$$A(e)=1-\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{x_i}{\mu}\right)^{1-e}\right)^{1/(1-e)} \tag{19}$$

where $e>0$ and $e\neq 1$.

If all $x_i\neq 0$, $A(1)$ is defined as $\lim_{e\to 1} A(e)$, what can be shown to be equal to

$$\frac{\mu-\mathrm{GM}(x_i)}{\mu}.$$

In this expression GM denotes the geometric mean.

Now,

$$A(e)'(X;\,Y)=-\frac{1}{1-e}\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{x_i}{\mu}\right)^{1-e}\right)^{e/(1-e)}\cdot$$

$$\left(\frac{(1-e)}{N\mu}\left(\frac{x_1}{\mu}\right)^{-e}-\frac{1-e}{\mu N(N-1)}\sum_{i=2}^{N}\left(\frac{x_i}{\mu}\right)^{-e}\right).$$

Then, if $e>1$, $A(e)'(X;\,Y)>0$ if and only if

$$x_1^{-e}>\frac{1}{N}\sum_{i=1}^{N}(x_i)^{-e}.$$

On the other hand, if $0<e<1$: $A(e)'(X;\,Y)>0$

$$\Leftrightarrow \quad x_1^{-e}<\frac{1}{N}\sum_{i=1}^{N}(x_i)^{-e}.$$

Again, the requirement to obtain a strictly positive directional derivative yields a relation involving a generalized mean. The special case $A(1)$ will involve a more familiar average.

$$A(1)=1-\frac{(x_1\ldots x_N)^{1/N}}{\mu} \tag{20}$$

$$A(1)'(X;\,Y)=-\frac{1}{N\mu}x_1^{(1-N)/N}(x_2\ldots x_N)^{1/N}$$

$$+\frac{1}{N(N-1)\mu}\sum_{i=2}^{N}x_i^{(1-N)/N}(x_1\ldots x_{i-1}x_{i+1}\ldots x_N)^{1/N}.$$

Hence: $A(1)'(X;\,Y)>0$

$$\Leftrightarrow \quad -\frac{1}{N\mu}x_1^{(1-N)/N}(x_2\ldots x_N)^{1/N}$$

$$>-\frac{1}{N(N-1)\mu}\sum_{i=2}^{N}x_i^{(1-N)/N}(x_1\ldots x_{i-1}x_{i+1}\ldots x_N)^{1/N}$$

$$\Leftrightarrow \quad x_1^{-1}<\frac{1}{N}\sum_{i=1}^{N}x_i^{-1}$$

$$\Leftrightarrow \quad x_1>\frac{N}{\sum_{i=1}^{N}\frac{1}{x_i}}=\mathrm{HM}(X).$$

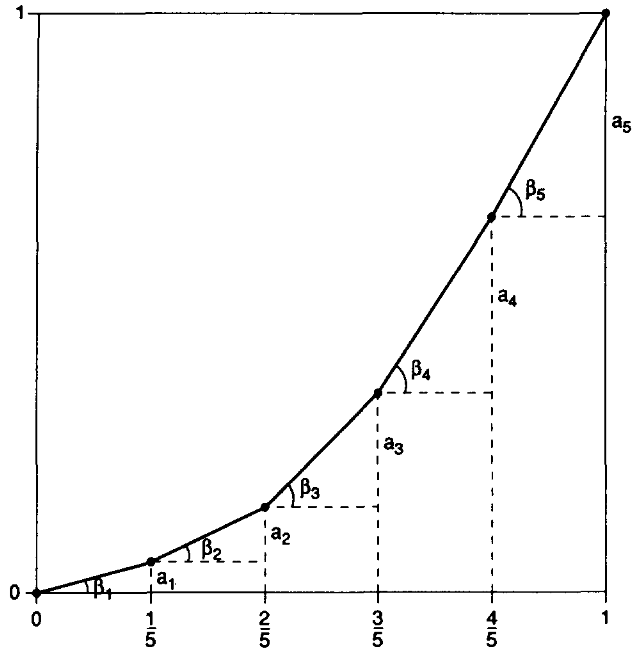

Fig. 1. The Lorenz curve of the vector $X=(1, 2, 4, 6, 7)$, showing that

$$\sin(\beta_i)=\frac{x_i}{\sqrt{x_i^2+\mu^2}}=\frac{a_i}{\sqrt{a_i^2+(1/N)^2}}.$$

(G) The length of the Lorenz curve: (LOR) (Dagum, 1980)

$$\mathrm{LOR}=\frac{1}{\mu N}\sum_{i=1}^{N}\sqrt{x_i^2+\mu^2} \tag{21}$$

$$\mathrm{LOR}'(X;Y)=\frac{1}{\mu N}\left(\frac{x_1}{\sqrt{x_1^2+\mu^2}}-\frac{1}{N-1}\sum_{i=2}^{N}\frac{x_i}{\sqrt{x_i^2+\mu^2}}\right).$$

Hence,

$$\mathrm{LOR}'(X;Y)>0$$

$$\Leftrightarrow\quad \frac{x_1}{\sqrt{x_1^2+\mu^2}}>\frac{1}{N-1}\sum_{i=2}^{N}\frac{x_i}{\sqrt{x_i^2+\mu^2}}$$

$$\Leftrightarrow\quad \sin(\omega_1)>\frac{1}{N}\sum_{i=1}^{N}\sin(\omega_i) \tag{22}$$

where

$$\sin(\omega_i)=\frac{x_i}{\sqrt{x_i^2+\mu^2}}=\frac{a_i}{\sqrt{a_i^2+(1/N)^2}}.$$

Here a special kind of average plays a role. Yet, it is easy to see where those sines come from. The Lorenz curve is built up by ranking the components in increasing order and connecting the points with coordinates

$$\left(\frac{i}{N},\sum_{j=1}^{i}a_j\right),\ i=1,\ldots,N. \tag{23}$$

Note that, as the components of X are now ordered, the main transfer does not anymore occur at the first component. We have denoted (see Fig. 1) the angles which the Lorenz segments make with the horizontal axis by the letter β. Then Fig. 1 clearly shows that $\sin(\beta_i)$ is nothing but the sine of the angle which the ith segment of the Lorenz curve makes with the horizontal axis.

We further note that the use of a trigonometric function in condition (22) is not as strange as it might seem at first. Indeed, Fig. 1 also shows that the familiar condition $x_i > \mu$ can also be expressed by a trigonometric function, as $x_i > \mu$ is equivalent to

$$\mathrm{tg}(\beta_i) > \frac{1}{N} \sum_{j=1}^{N} \mathrm{tg}(\beta_j). \tag{24}$$

We conclude this section by observing that indeed all kinds of averages play a role in this description of transfer sensitivity.

5. COMMENTS

(1) It is well known (Hardy *et al.*, 1952, 2.9) that for $-\infty < r < s < +\infty$

$$\left(\frac{1}{N}\sum_{i=1}^{N} x_i^r\right)^{1/r} < \left(\frac{1}{N}\sum_{i=1}^{N} x_i^s\right)^{1/s}$$

where the x_i are positive and not all are equal. In particular, for $r=1$ we have the arithmetic mean, for $r \to 0$ the expression tends to the geometric mean (GM) and for $r=-1$ we have the harmonic mean (HM). This collection of averages leads to a battery of sensitivities. Indeed, if, for example, $x_1 > \mu$, $V'(X; Y) > 0$, but, as $\mu > \mathrm{GM}$, $x_1 > \mathrm{GM}(X)$, hence also $\mathrm{Th}'(X; Y) > 0$.

(2) The opposite, of course, does not hold. Consider, for example, the following example: $X = (6, 2, 4, 5, 16)$; $\mu(X) = 6.6$, $\mathrm{GM}(X) = 5.2$. Taking now $x_1 = 6$, for the main transfer we find that

$$\mathrm{Th}'(X; Y) = 0.005345 > 0$$

but

$$V'(X; Y) = -0.022727 < 0.$$

These inequalities hold for every x_1 satisfying $\mathrm{GM}(X) = 5.2 < x_1 < \mu(X) = 6.6$.

(3) The case of LOR, involving sine functions, can also be compared with other measures. Indeed, if $x_1 > \mu$ then also

$$\frac{x_1}{\sqrt{x_1^2 + \mu^2}} > \frac{1}{N} \sum_{i=1}^{N} \frac{x_i}{\sqrt{x_i^2 + \mu^2}} \tag{25}$$

$$\Leftrightarrow \quad \sin(\omega_1) > \frac{1}{N} \sum_{i=1}^{N} \sin(\omega_i)$$

where

$$\sin(\omega_i) = \frac{x_i}{\sqrt{x_i^2 + \mu^2}} = \frac{a_i}{\sqrt{a_i^2 + (1/N)^2}}.$$

This follows from the fact that if $x_1 > \mu > 0$

$$\frac{x_1}{\sqrt{x_1^2 + \mu^2}} > \frac{1}{\sqrt{2}}.$$

Further $(\mu, \ldots, \mu) \prec (x_1, x_2, \ldots, x_N)$ and as the function

$$t \to \frac{t}{\sqrt{t^2 + \mu^2}}$$

is concave, it follows by (Hardy *et al.*, 1952, p. 89) that

$$\frac{1}{\sqrt{2}}>\frac{1}{N}\sum_{i=1}^{N}\frac{x_i}{\sqrt{x_i^2+\mu^2}}.$$

Note however that this implication holds only in one direction!

(4) The transfer principle (T_M) is independent of the classical transfer principles (T). Consider for example the relative mean deviation, denoted as D:

$$D=\frac{\frac{1}{N}\sum_{i=1}^{N}|x_i-\mu|}{2\mu}. \tag{26}$$

This function is not a proper, i.e. type I, concentration measure (Dalton, 1920). Yet, if $x_1>\mu$, $D'(X;Y)>0$. This can be seen as follows:

$$D'(X;Y)>0$$

$$\Leftrightarrow \quad \frac{1}{2N\mu}\left(\frac{x_1-\mu}{\sqrt{(x_1-\mu)^2}}-\frac{1}{N-1}\sum_{i=2}^{N}\frac{x_i-\mu}{\sqrt{(x_i-\mu)^2}}\right)>0$$

$$\Leftrightarrow \quad x_1>\frac{1}{N}\sum_{i=1}^{N}x_i=\mu.$$

(5) We have done similar calculations, and obtained similar results, for a number of diversity measures (such as Simpson's index). In this case conditions are obtained for the directional derivative to be negative.

6. CONCLUSION

It is shown how different inequality measures behave with respect to the directional derivative we have introduced. We have shown that different averages, such as the arithmetic mean, the median, the harmonic mean and the geometric mean, but also more general averages, play an essential role in studying sensitivities to transfers. We conclude that the use of this directional derivative introduces a battery of sensitivities in the class of inequality measures. This will help the scientist to choose between otherwise acceptable measures.

REFERENCES

Allison, P. D. (1978). Measures of inequality. *American Sociological Review, 43*, 865–880.

Atkinson, A. B. (1970). On the measurement of inequality. *Journal of Economic Theory, 2*, 244–263.

Burrell, Q. L. (1991). The Bradford distribution and the Gini index. *Scientometrics, 21*, 181–194.

Dagum, C. (1980). The generation and distribution of income, the Lorenz curve and the Gini ratio. *Economie Appliquée, 33*, 327–367.

Dalton, H. (1920). The measurement of the inequality of incomes. *The Economic Journal, 30*, 348–361.

Egghe, L. (1987). Pratt's measure for some bibliometric distribution and its relation with the 80/20 rule. *Journal of the American Society for Information Science, 38*, 288–297.

Egghe, L. (1994). A theory of continuous rates and applications to the theory of growth and obsolescence rates. *Information Processing & Management, 30*, 279–292.

Egghe, L., & Rousseau, R. (1990). Elements of concentration theory. In L. Egghe & R. Rousseau (Eds), *Informetrics 89/90* (pp. 97–137). Amsterdam: Elsevier.

Egghe, L., & Rousseau, R. (1991). Transfer principles and a classification of concentration measures. *Journal of the American Society for Information Science, 42*, 479–489.

Hardy, G., Littlewood, J. E., & Pólya, G. (1952). *Inequalities* (2nd edn). Cambridge: Cambridge University Press.

Herdan, G. (1966). *The advanced theory of language as choice and chance.* Berlin: Springer.

Magurran, A. E. (1991). *Ecological diversity and its measurement.* London: Chapman & Hall.

Protter, M. H., & Morrey, C. B. (1977). *A first course in real analysis*. New York: Springer.
Rousseau, R. (1992). Concentration and diversity measures: Dependence on the number of classes. *Belgian Journal of Operations Research, Statistics and Computer Science, 32*, 99–126.
Rousseau, R. (1993). Measuring concentration: Sampling design issues, as illustrated by the case of perfectly stratified samples. *Scientometrics, 28*(1), 3–14.
Rousseau, R., & Van Hecke, P. (1993). Introduction of a species does not necessarily increase diversity. *Coenoses, 8*, 39–40.
Sen, A. (1973). *On economic inequality.* Oxford: Clarendon Press.
Theil, H. (1967). *Economics and information theory.* Amsterdam: North-Holland.
White, M. J. (1986). Segregation and diversity measures in population distribution. *Population Index, 52*, 198–221.