Physica A 490 (2018) 1277–1294

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

On the motion of substance in a channel of a network and human migration

Nikolay K. Vitanov[a,b,*], Kaloyan N. Vitanov[a]

[a] Institute of Mechanics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Bl. 4, 1113 Sofia, Bulgaria
[b] Max-Planck Institute for the Physics of Complex Systems, Nöthnitzerstr. 38, 01187 Dresden, Germany

HIGHLIGHTS

- A model is discussed for motion of substance in a channel made by nodes of a network.
- Heavy tailed distributions are obtained for the amount of substance in the nodes.
- The model is applied to human migration in a chain of countries.
- Conditions for concentration of migrants in a country of the channel are obtained.
- Zipf distribution for number of migrants is obtained.

ARTICLE INFO

Article history:
Received 26 February 2017
Received in revised form 16 May 2017
Available online 1 September 2017

Keywords:
Network channels
Network flows
Waring distribution
Social networks
Human migration

ABSTRACT

We model the motion of a substance in a channel of a network that consists of chain of (i) nodes of the network and (ii) edges that connect the nodes and form the way for motion of the substance. The nodes of the channel can have different “leakage”, i.e., some amount of the substance can leave the channel at a node and the rate of leaving can be different for the different nodes of the channel. The nodes close to the end of the channel for some (design or other) reason may be more “attractive” for the substance in comparison to the nodes around the incoming node of the channel. We discuss channels containing infinite or finite number of nodes. The main outcome of the model is the distribution of the substance along the nodes. Two regimes of functioning of the channels are studied: stationary regime and non-stationary regime. The distribution of the substance along the nodes of the channel for the case of stationary regime is a distribution with a very long tail that contains as particular case the Waring distribution (for channel with infinite number of nodes) or the truncated Waring distribution (for channel with finite number of nodes). In the non-stationary regime of functioning of the channel one observes an exponential increase or exponential decrease of the amount of substance in the nodes. However the asymptotic distribution of the substance among the nodes of the channel in this regime remains stationary.

The studied model is applied to the case of migration of humans through a migration channel consisting of chain of countries. In this case the model accounts for the number of migrants entering the channel through the first country of the channel; permeability of the borders between the countries; possible large attractiveness of some countries of the channel; possibility for migrants to obtain permission to reside in a country of the channel. The main outcome of the model is the distribution of migrants along the countries of the channel. We discuss the conditions for concentration of migrants in selected country of the channel. Finally two scenarios of changes of conditions of the functioning of the channel are

---

[*] Corresponding author at: Institute of Mechanics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Bl. 4, 1113 Sofia, Bulgaria.
E-mail address: vitanov@imbm.bas.bg (N.K. Vitanov).

http://dx.doi.org/10.1016/j.physa.2017.08.038

discussed. It is shown that from the point of view of decreasing of the number of migrants in the countries of the channel it is more effective to concentrate efforts on preventing the entrance of migrants in the first country of the channel when compared to concentration of efforts on decrease of permeability of the borders between the countries of the channel.

1. Introduction

Research on network flows has some of its roots in the studies on transportation problems, e.g., in the developing of minimal cost transportation models. This research topic was established in 1960's especially after the publishing the book of Ford and Fulkerson [1]. At the beginning of the research the problems of interest have been, e.g., how by minimal number of individuals to meet a fixed schedule of tasks; minimal cost flow problems; or possible maximal flows in a network. In course of the years the area of problems connected to network flows has increased very much. Nowadays one uses the methodology from the theory of network flows [2,3] to solve problems connected to: (i) shortest path finding, (ii) just in time scheduling, (iii) facility layout and location, (iv) project management (determining minimum project duration), (v) optimal electronic route guidance in urban traffic networks [4], (vi) self-organizing network flows, (vii) modeling and optimization of scalar flows in networks [5], (viii) memory effects [6], (ix) isoform identification of RNA [7], etc. (just some other examples are [8–14]).

Below we shall consider a specific network flow problem: motion of a substance through a network channel in presence of possibility for "leakage" in the nodes of the channel (loss of substance or usage of a part of substance in some process). In addition we shall assume that the substance may have preference for some of the nodes of the channel (e.g., the channel may be designed in such a way that the substance tends to concentrate in some of the nodes). This feature will allow us to use the model for study of motions of animals or humans. We note that the discussed model contains also the particular case when there is no preference of the substance with respect to the nodes of the channel. The obvious application of the model is for the flow of some non-living substance through a channel with use of part of substance for some industrial process in the nodes of the channel. We shall show that the model has more applications by another illustration: modeling large human migration flows. The large flows allow continuous modeling as in this case the discrete quantities can be approximated by continuous ones. Imagine a chain of countries that form a migration channel. The first country of this channel may have a sea border and the migrants may come to this country (called below incoming country or entry country of the channel) through this sea border. In addition one or several countries of the channel may be preferred by migrants. Such choice of an illustration of the model is motivated also by actuality of the problem of human migration [15]. Indeed the study of international migration becomes very actual after the large migration flows directed to Europe in 2015. Much efforts are invested also in the study of internal migration in order to understand this migration and to make projection of the migration flows that may be very important for taking decisions about economic development of regions of a country [16–24]. Human migration models are of interest also for applied mathematics as they can be classified as probability models (exponential model, Poisson model, multinomial model, Markov chain models of migration [25–36]) or deterministic models (e.g., gravity model of migration [37]). Human migration is closely connected to migration networks [38,39], to ideological struggles [40,41] and to waves and statistical distributions in population systems [42–45].

The paper is organized as follows. In Section 2 we discuss a model for motion of substance in a channel containing infinite number of nodes. Two regimes of functioning of the channel are studied: stationary regime and non-stationary regime. Statistical distributions of the amount of substance in the nodes of the channel are obtained. A particular case of the distribution for the stationary regime of functioning of the channel is the Waring distribution. Section 3 is devoted to the case of channel containing finite number of nodes. This case is of interest for the problem of human migration. The distribution of substance for stationary regime of functioning of such a channel is given by a distribution that is a generalization of the truncated Waring distribution. In Section 4 we discuss several outcomes of the model for the case of human migration channel of finite length, e.g., the possibility of concentration of migrants in the last node of the channel (the final destination country). Several concluding remarks are summarized in Section 5.

2. Channel containing infinite number of nodes

We consider a channel consisting of a chain of nodes of a network. The nodes are connected by edges and each node is connected only to the two neighboring nodes of the channel with the exception of the first and the last node of the channel that are connected only to the neighboring node. We study a model of the motion of substance through such a channel which is an extension of the model discussed in [46] and [47]. We consider each node as a cell (box), i.e., we consider an array of infinite number of cells indexed in succession by non-negative integers. The first cell has index 0 and the last cell has index

N (in the case discussed here $N = \infty$). We assume that an amount x of some substance is distributed among the cells and this substance can move from one cell to another cell. Let x_i be the amount of the substance in the ith cell. Then

$$x = \sum_{i=0}^{\infty} x_i \tag{2.1}$$

The fractions $y_i = x_i/x$ can be considered as probability values of distribution of a discrete random variable ζ

$$y_i = p(\zeta = i),\ i = 0, 1, \ldots \tag{2.2}$$

The content x_i of any cell may change due to the following 3 processes:

1. Some amount s of the substance x enters the system of cells from the external environment through the 0th cell;
2. Amount f_i from x_i is transferred from the ith cell into the $i+1$th cell;
3. Amount g_i from x_i leaks out the ith cell into the external environment.

We assume that the process of the motion of the substance is continuous in the time. Then the process can be modeled mathematically by the system of ordinary differential equations:

$$\frac{dx_0}{dt} = s - f_0 - g_0;$$
$$\frac{dx_i}{dt} = f_{i-1} - f_i - g_i,\ i = 1, 2, \ldots . \tag{2.3}$$

There are two regimes of functioning of the channel: stationary regime and non-stationary regime.

2.1. Stationary regime of functioning of the channel

In the stationary regime of the functioning of the channel $\frac{dx_i}{dt} = 0, i = 0, 1, \ldots$. Let us mark the quantities for the stationary case with *. Then from Eqs. (2.3) one obtains

$$f_0^* = s^* - g_0^*;\ \ f_i^* = f_{i-1}^* - g_i. \tag{2.4}$$

This result can be written also as

$$f_i^* = s^* - \sum_{j=0}^{i} g_j^* \tag{2.5}$$

Hence for the stationary case the situation in the channel is determined by the quantities s^* and $g_j^*, j = 0, 1, \ldots$. In this paper we shall assume the following forms of the amount of the moving substances in Eqs. (2.3) ($\alpha, \beta, \gamma_i, \sigma$ are constants)

$$s = \sigma x_0 = \sigma_0;\ \ \sigma_0 > 0$$
$$f_i = (\alpha_i + \beta_i i)x_i;\ \ \ \alpha_i > 0,\ \beta_i \geq 0$$
$$g_i = \gamma_i x_i;\ \ \ \gamma_i \geq 0 \rightarrow \text{non-uniform leakage in the nodes} \tag{2.6}$$

The rules (2.6) differ from the rules in [46] in 3 points:

1. s is proportional to the substance in the 0th node (the amount of this substance is x_0). In [46] s is proportional to the amount x of the substance in the entire channel $x = \sum_{i=0}^{N} x_i$;
2. Leakage rate γ_i is different for the different nodes. In [46] the leakage rate is constant and equal to γ for all nodes of the channel (i.e., there is uniform leakage in the nodes).
3. Parameters α_i and β_i are different for the different cells. In [46] these parameters are the same for all cells of the channel. This may lead to cumulative advantage of higher nodes which is not guaranteed in the model discussed in this article.

Substitution of Eqs. (2.6) in Eqs. (2.3) leads to the relationships

$$\frac{dx_0}{dt} = \sigma_0 x_0 - \alpha_0 x_0 - \gamma_0 x_0;$$
$$\frac{dx_i}{dt} = [\alpha_{i-1} + (i-1)\beta_{i-1}]x_{i-1} - (\alpha_i + i\beta_i + \gamma_i)x_i;\ \ \ i = 1, 2, \ldots \tag{2.7}$$

As we shall consider the stationary regime of functioning of the channel then from the first of the Eqs. (2.7) it follows that $\sigma_0 = \alpha_0 + \gamma_0$. This means that x_0 (the amount of the substance in the 0th cell of the channel) is free parameter. In this case the solution of Eqs. (2.7) is

$$x_i = x_i^* + \sum_{j=0}^{i} b_{ij} \exp[-(\alpha_j + j\beta_j + \gamma_j)t] \tag{2.8}$$

where x_i^* is the stationary part of the solution. For x_i^* one obtains the relationship

$$x_i^* = \frac{\alpha_{i-1} + (i-1)\beta_{i-1}}{\alpha_i + i\beta_i + \gamma_i} x_{i-1}^*, \; i = 1, 2, \dots \tag{2.9}$$

The corresponding relationships for the coefficients b_{ij} are ($i = 1, \dots$):

$$b_{ij} = \frac{\alpha_{i-1} + (i-1)\beta_{i-1}}{(\alpha_i - \alpha_j) + (i\beta_i - j\beta_j) + (\gamma_i - \gamma_j)} b_{i-1,j}, \; j = 0, 1, \dots, i-1 \tag{2.10}$$

From Eq. (2.9) one obtains

$$x_i^* = \frac{\prod_{j=0}^{i-1}[\alpha_{i-j-1} + (i-j-1)\beta_{i-j-1}]}{\prod_{j=0}^{i-1} \alpha_{i-j} + (i-j)\beta_{i-j} + \gamma_{i-j}} x_0^* \tag{2.11}$$

The form of the corresponding stationary distribution $y_i^* = x_i^*/x^*$ (where x^* is the amount of the substance in all of the cells of the channel) is

$$y_i^* = \frac{\prod_{j=0}^{i-1}[\alpha_{i-j-1} + (i-j-1)\beta_{i-j-1}]}{\prod_{j=0}^{i-1} \alpha_{i-j} + (i-j)\beta_{i-j} + \gamma_{i-j}} y_0^* \tag{2.12}$$

To the best of our knowledge the distribution presented by Eq. (2.12) was not discussed by other authors. Figs. 1–3 show the shape of this long tail distribution as well as the dependence of the shape of the distribution on the parameters of distribution. Let us show that the distribution (2.12) contains as particular cases several famous distributions such as Waring distribution, Zipf distribution, and Yule–Simon distribution. In order to do this we consider the particular case when $\beta_i \neq 0$ and write x_i from Eq. (2.11) as follows

$$x_i^* = \frac{\prod_{j=0}^{i-1} b_{i-j}[k_{i-j-1} + (i-j-1)]}{\prod_{j=0}^{i-1}[k_{i-j} + a_{i-j} + (i-j)]} x_0^* \tag{2.13}$$

where $k_i = \alpha_i/\beta_i$; $a_i = \gamma_i/\beta_i$; $b_i = \beta_{i-1}/\beta_i$. The form of the corresponding stationary distribution $y_i^* = x_i^*/x^*$ is

$$y_i^* = \frac{\prod_{j=0}^{i-1} b_{i-j}[k_{i-j-1} + (i-j-1)]}{\prod_{j=0}^{i-1}[k_{i-j} + a_{i-j} + (i-j)]} y_0^* \tag{2.14}$$

Let us now consider the particular case where $\alpha_i = \alpha$ and $\beta_i = \beta$ for $i = 0, 1, 2, \dots$. Then from Eq. (2.13) one obtains

$$x_i^* = \frac{[k + (i-1)]!}{(k-1)! \prod_{j=1}^{i}(k + j + a_j)} x_0^* \tag{2.15}$$

where $k = \alpha/\beta$ and $a_j = \gamma_j/\beta$. The form of the corresponding stationary distribution $y_i^* = x_i^*/x^*$ is

$$y_i^* = \frac{[k + (i-1)]!}{(k-1)! \prod_{j=1}^{i}(k + j + a_j)} y_0^* \tag{2.16}$$

Let us consider the particular case where $a_0 = \dots = a_N = a$. In this case the distribution from Eq. (2.16) is reduced to the distribution:

$$P(\zeta = i) = P(\zeta = 0)\frac{(k-1)^{[i]}}{(a+k)^{[i]}}; \;\; k^{[i]} = \frac{(k+i)!}{k!}; \; i = 1, 2, \dots \tag{2.17}$$

$P(\zeta = 0) = y_0^* = x_0^*/x^*$ is the percentage of substance that is located in the first cell of the channel. Let this percentage be

$$y_0^* = \frac{a}{a+k} \tag{2.18}$$

The case described by Eq. (2.17) corresponds to the situation where the amount of substance in the first cell is proportional of the amount of substance in the entire channel (self-reproduction property of the substance). In this case Eq. (2.16) is reduced to the distribution:

$$P(\zeta = i) = \frac{a}{a+k}\frac{(k-1)^{[i]}}{(a+k)^{[i]}}; \;\; k^{[i]} = \frac{(k+i)!}{k!}; \; i = 1, 2, \dots \tag{2.19}$$

Let us denote $\rho = a$ and $k = l$. The distribution (2.19) is exactly the Waring distribution (probability distribution of non-negative integers named after Edward Waring—the 6th Lucasian professor of Mathematics in Cambridge from the 18th century) [48–50] —Figs. 4, 5.

$$p_l = \rho \frac{\alpha_{(l)}}{(\rho + \alpha)_{(l+1)}}; \; \alpha_{(l)} = \alpha(\alpha+1)\dots(\alpha+l-1) \tag{2.20}$$

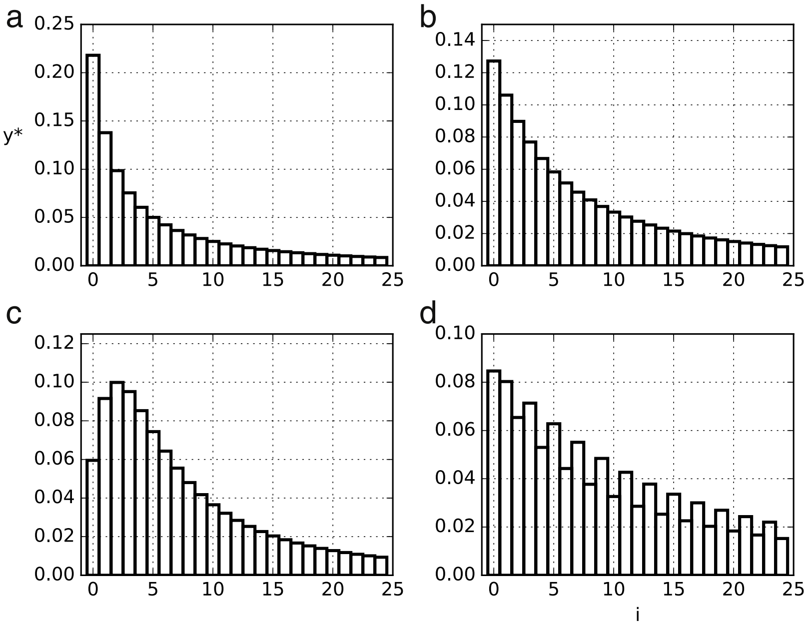

Fig. 1. Distribution (2.12) for various values of the parameters α_i, β_i, and γ_i. Values of the parameters β_i and γ_i are fixed: $\beta_i = 0.01$, $\gamma_i = 0.01$. Figures show the influence of different choices of the parameters α_i on the shape of the distribution. (a) $\alpha_i = 0.05 + 0.1i/(i+10)$. The long tail of the distribution is clearly visible. (b) $\alpha = 0.01$. This figure shows a shape of the distribution for fixed values of all three parameters of the distribution. (c) $\alpha_i = 0.2 - 0.18i/(i+1)$. Decreasing values of α_i with increasing i can lead to changes in the shape of the distribution. The long tail of the distribution continues to exist. (d) $\alpha_i = 0.2 + 0.1(-1)^i i/(i+10)$. The shape of distribution is sensitive to differences of the values of parameters α along the channel.

Waring distribution may be written also as follows

$$\begin{aligned} p_0 &= \rho \frac{\alpha_{(0)}}{(\rho+\alpha)_{(1)}} = \frac{\rho}{\alpha+\rho} \\ p_l &= \frac{\alpha+(l-1)}{\alpha+\rho+l} p_{l-1}. \end{aligned} \tag{2.21}$$

The mean μ (the expected value) of the Waring distribution is

$$\mu = \frac{\alpha}{\rho - 1} \text{ if } \rho > 1 \tag{2.22}$$

The variance of the Waring distribution is

$$V = \frac{\alpha\rho(\alpha+\rho-1)}{(\rho-1)^2(\rho-2)} \text{ if } \rho > 2 \tag{2.23}$$

ρ is called the tail parameter as it controls the tail of the Waring distribution. Waring distribution contains various distributions as particular cases. For an example for large values of l the Waring distribution has the behavior

$$p_l \approx \frac{1}{l^{(1+\rho)}}. \tag{2.24}$$

which is the same as the frequency form of the Zipf distribution [51]. If $\alpha \to 0$ the Waring distribution is reduced to the Yule–Simon distribution [52]

$$p(\zeta = l \mid \zeta > 0) = \rho B(\rho+1, l) \tag{2.25}$$

where B is the beta-function.

Waring distribution has been considered by Irwin [53] to be a very suitable theoretical form for description of biological distributions with very long tail. In the following years Waring distribution and truncated Waring distribution (that will be mentioned below in the text) have found interesting applications, e.g. in quantitative linguistics [54], bibliometrics [46,55–57], and models of migration dynamics [47].

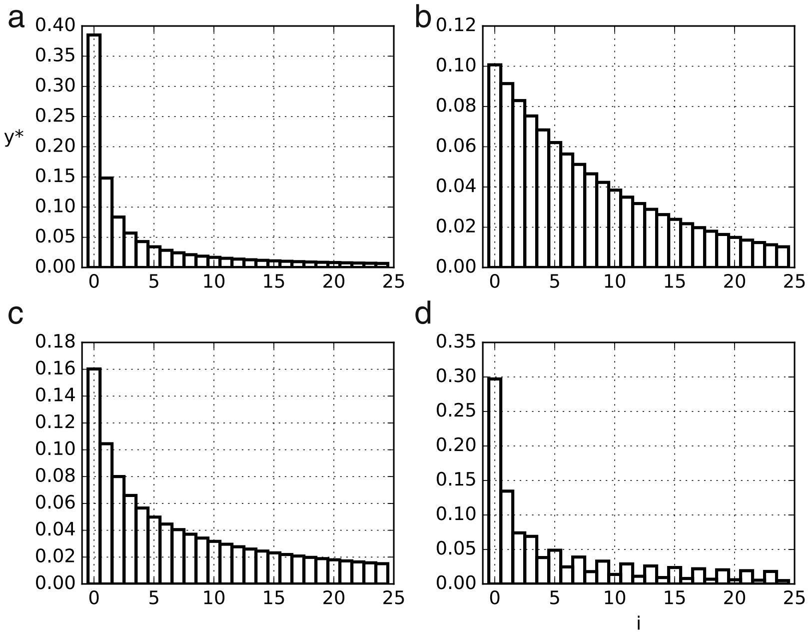

Fig. 2. Distribution (2.12) for various values of the parameters α_i, β_i, and γ_i. Values of the parameters α_i and γ_i are fixed: $\alpha_i = 0.1$, $\gamma_i = 0.01$. Figures show the influence of different choices of the parameters β_i on the shape of the distribution. (a) $\beta_i = 0.1 + 0.1i/(i + 10)$. (b) $\beta = 0.01$. This is another combination of constant values of parameters in comparison to Fig. 1(b). The long tail of the distribution is preserved. (c) $\beta_i = 0.05 - 0.04i/(i + 5)$. This combination of parameters of the distribution does not lead to change of the shape of the distribution. The long tail of the distribution continues to exist. (d) $\beta_i = 0.12 + 0.1(-1)^i i/(i + 10)$. The shape of distribution is sensitive to differences of the values of parameters β along the channel. The large probability y_0^* connected to the 0th node of the channel limits the changes of the values of distribution for large i.

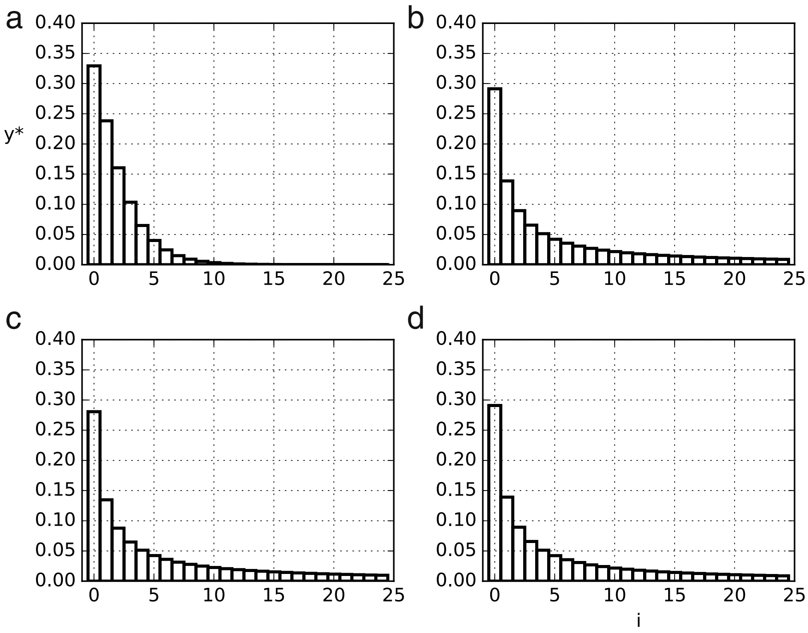

Fig. 3. Distribution (2.12) for various values of the parameters α_i, β_i, and γ_i. Values of the parameters α_i and γ_i are fixed: $\alpha_i = 0.1$, $\beta_i = 0.1$. Figures show the influence of different choices of the parameters γ_i on the shape of the distribution. (a) $\gamma_i = 0.1 + 0.2i/(i + 10)$. The increase of the values of γ leads to a fast decrease of the probability y_i^*. (b) $\gamma = 0.01$. The long tail of the distribution is clearly visible. (c) $\gamma_i = 0.01 - 0.01i/(i + 5)$. The decrease of γ_i with increasing i preserves the long tail of the distribution. (d) $\gamma_i = 0.1 + 0.1(-1)^i i/(i + 10)$. The shape of distribution has smaller sensitivity to differences of the values of parameters γ along the channel as compared to analogous changes of the parameters α_i and β_i shown in Figs. 1(d) and 2(d).

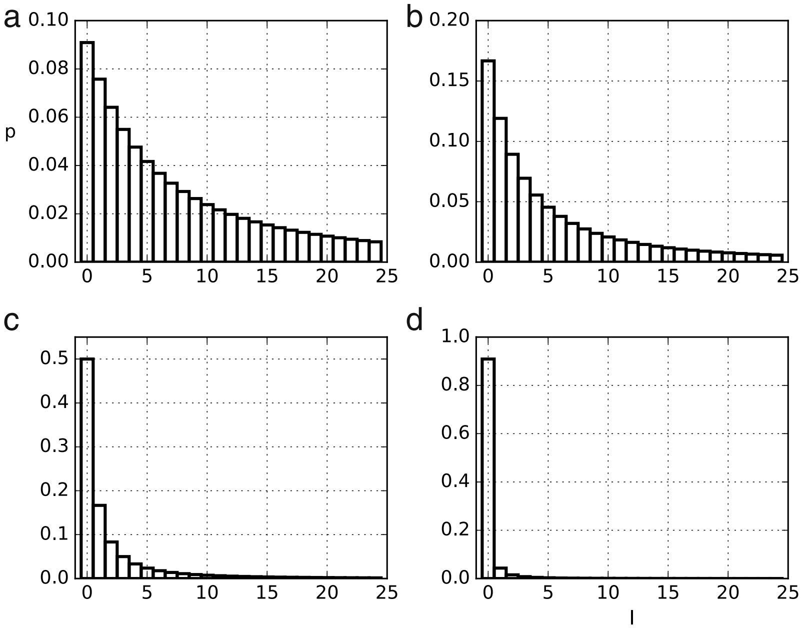

Fig. 4. Waring distribution. The Waring distribution (2.21) depends on two parameters ρ and α. The more general distribution (2.12) depends on 3 parameters. The figure shows the influence of the parameter α on the shape of the distribution when the value of the parameter ρ is fixed. (a) $\alpha = 10$, $\rho = 1$. The long tail of the Waring distribution is clearly visible. (b) $\alpha = 5$, $\rho = 1$. The value of ρ is the same as in (a). Thus the influence of decrease of parameter α on the Waring distribution is to increase the probability p_l for small values of l and to decrease probabilities connected to the tail of the distribution. (c) $\alpha = 1$, $\rho = 1$. Further decrease of the value of α leads to continuation of the changes in the shape of the Waring distribution—probabilities p_l for small values of l increase at the expense of decreasing probabilities connected to the tail of the distribution. (d) $\alpha = 0.1$, $\rho = 1$. Further decrease of the value of ρ leads almost to vanishing of the long tail of the distribution. We note that the changes in the shape of the Waring distributions are smaller in comparison to the possible changes of the more general distribution (2.12) shown in Figs. 1–3.

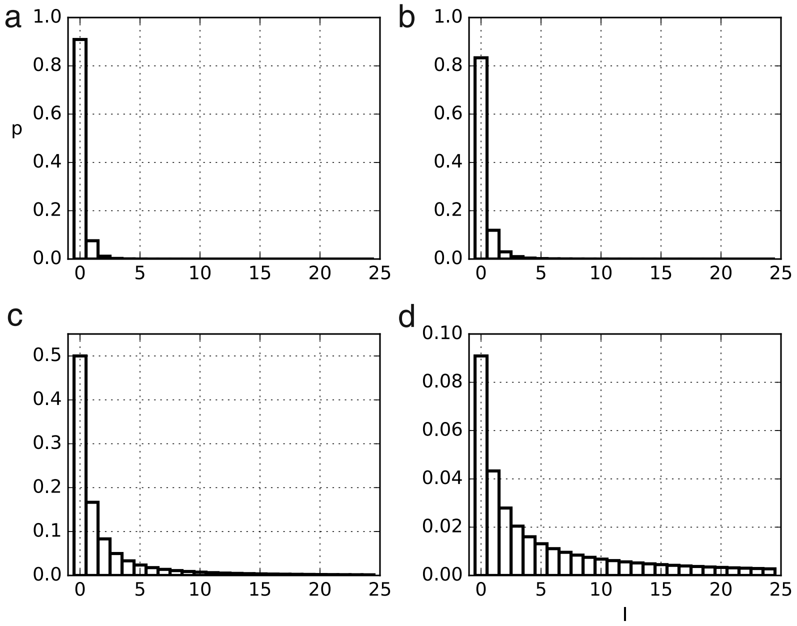

Fig. 5. Waring distribution. Figure shows the influence of the parameter ρ on the shape of the distribution when the value of the parameter α is fixed. (a) $\alpha = 1$, $\rho = 10$. The long tail of the Waring distribution is almost nonexistent. (b) $\alpha = 1$, $\rho = 5$. The value of ρ is the same as in (a). Thus the influence of decrease of the value of parameter ρ on the Waring distribution is to decrease the probability p_l for smaller values of l and to increase probabilities connected to the tail of the distribution. (c) $\alpha = 1$, $\rho = 1$. Further decrease of the value of ρ leads to continuation of the changes in the shape of the Waring distribution—probabilities p_l for small values of l decrease at the expense of increasing probabilities connected to the tail of the distribution. We note that the values of the parameters in Fig. 2(c) are the same as the values of the parameters in Fig. 1(c). This leads to additional information about the influence of parameters α and ρ on the shape of the Waring distribution. (d) $\alpha = 1$, $\rho = 0.1$. Further decrease of the value of ρ leads almost to clear visibility of the long tail of the distribution.

2.2. Non-stationary regime of functioning of the channel

In the nonstationary case $dx_0/dt \neq 0$. In this case the solution of the first equation of the system of Eqs. (2.7) is

$$x_0 = b_{00} \exp[(\sigma_0 - \alpha_0 - \gamma_0)t] \tag{2.26}$$

where b_{00} is a constant of integration. x_i must be obtained by solution of the corresponding Eqs. (2.7). The form of x_i is

$$x_i = \sum_{j=0}^{i} b_{ij} \exp[-(\alpha_j + j\beta_j + \gamma_j - \sigma_j)t] \tag{2.27}$$

The solution of the system of Eqs. (2.7) is (2.27) where $\sigma_i = 0, i = 1, \dots, (\sigma_0 = \sigma)$,

$$b_{ij} = \frac{\alpha_{i-1} + (i-1)\beta_{i-1}}{(\alpha_i - \alpha_j) + (i\beta_i - j\beta_j) + (\gamma_i - \gamma_j)} b_{i-1,j};\ i = 1, \dots; j = 1, \dots, i-1, \tag{2.28}$$

and b_{ii} are determined from the initial conditions in the cells of the channel. The asymptotic solution ($t \to \infty$) is

$$x_i^a = b_{i0} \exp[(\sigma_0 - \alpha_0 - \gamma_0)t] \tag{2.29}$$

This means that the asymptotic distribution $y_i^a = x_i^a/x^a$ is stationary

$$y_i^a = \frac{b_{i0}}{\sum_{j=0}^{\infty} b_{j0}} \tag{2.30}$$

regardless of the fact that the amount of substance in the two cells may increase or decrease exponentially. The explicit form of this distribution is

$$\begin{aligned} y_0^a &= \frac{1}{1 + \sum_{i=1}^{\infty} \prod_{k=1}^{i} \frac{\alpha_{k-1}+(k-1)\beta_{k-1}}{(\alpha_k-\alpha_0)+k\beta_k+(\gamma_k-\gamma_0)}} \\ y_i^a &= \frac{\prod_{k=1}^{i} \frac{\alpha_{k-1}+(k-1)\beta_{k-1}}{(\alpha_k-\alpha_0)+k\beta_k+(\gamma_k-\gamma_0)}}{\sum_{i=0}^{\infty} \prod_{k=1}^{i} \frac{\alpha_{k-1}+(k-1)\beta_{k-1}}{(\alpha_k-\alpha_0)+k\beta_k+(\gamma_k-\gamma_0)}}, i = 1, \dots, \end{aligned} \tag{2.31}$$

3. Channel containing finite number of nodes

Finite size channels are very interesting from the point of view of the applications of the theory, e.g., to migrant flows. Let us consider a channel consisting of $N + 1$ nodes (cells) and corresponding edges. The nodes are indexed in succession by non-negative integers, i.e., the first cell has index 0 and the last cell has index N. In this case the total amount of substance in the channel is

$$x = \sum_{i=0}^{N} x_i \tag{3.1}$$

The fractions $y_i = x_i/x$ can be considered as probability values of distribution of a discrete random variable ζ

$$y_i = p(\zeta = i),\ i = 0, 1, \dots, N \tag{3.2}$$

The mathematical model of the finite channel is as follows:

$$\begin{aligned} \frac{dx_0}{dt} &= s - f_0 - g_0; \\ \frac{dx_i}{dt} &= f_{i-1} - f_i - g_i,\ i = 1, 2, \dots, N-1 \\ \frac{dx_N}{dt} &= f_{N-1} - g_N. \end{aligned} \tag{3.3}$$

The relationships for the amount of the moving substances are the same as in the case of infinite channel:

$$\begin{aligned} s &= \sigma x_0 = \sigma_0 x_0;\quad \sigma = \sigma_0 > 0 \\ f_i &= (\alpha_i + \beta_i i)x_i;\quad \alpha_i > 0,\ \beta_i \geq 0 \\ g_i &= \gamma_i x_i;\quad \gamma_i \geq 0 \to \text{non-uniform leakage in the nodes} \end{aligned} \tag{3.4}$$

Substitution of Eqs. (3.4) in Eqs. (3.3) leads to the relationships

$$\begin{aligned}\frac{dx_0}{dt} &= \sigma_0 x_0 - \alpha_0 x_0 - \gamma_0 x_0;\\ \frac{dx_i}{dt} &= [\alpha_{i-1} + (i-1)\beta_{i-1}]x_{i-1} - (\alpha_i + i\beta_i + \gamma_i)x_i,\ i = 1, 2, \dots, N-1\\ \frac{dx_N}{dt} &= [\alpha_{N-1} + (N-1)\beta_{N-1}]x_{N-1} - \gamma_N x_N\end{aligned} \qquad (3.5)$$

3.1. Stationary regime of functioning of the channel

In the stationary regime of functioning of the channel $\frac{dx_0}{dt} = 0$, i.e. $\sigma_0 = \alpha_0 + \gamma_0$. In this case the system of Eqs. (3.5) has a stationary solution with a free parameter x_0. This solution is

$$\begin{aligned}x_i^* &= \frac{\alpha_{i-1} + (i-1)\beta_{i-1}}{\alpha_i + i\beta_i + \gamma_i} x_{i-1}^*,\ i = 1, 2, \dots, N-1\\ x_N^* &= \frac{\alpha_{N-1} + (N-1)\beta_{N-1}}{\gamma_N} x_{N-1}^*.\end{aligned} \qquad (3.6)$$

The solution of Eqs. (3.5) is

$$x_i = x_i^* + \sum_{j=0}^{i} b_{ij} \exp[-(\alpha_j + j\beta_j + \gamma_j)t] \qquad (3.7)$$

The substitution of Eq. (3.7) in Eqs. (3.5) leads to the following relationships for the coefficients b_{ij} ($\alpha_N = \beta_N = 0$ as there is no $N+1$st node(cell) where the substance can move from the Nth node (cell))

$$\begin{aligned}b_{ij} &= \frac{\alpha_{i-1} + (i-1)\beta_{i-1}}{(\alpha_i - \alpha_j) + (i\beta_i - j\beta_j) + (\gamma_i - \gamma_j)} b_{i-1,j};\\ &\quad i = 1, \dots, N-1;\ \ j = 0, \dots, i;\\ b_{Nj} &= \frac{\alpha_{N-1} + (N-1)\beta_{N-1}}{\gamma_N - \gamma_j - \alpha_j - j\beta_j} b_{N-1,j},\ j = 0, \dots, N-1.\end{aligned} \qquad (3.8)$$

b_{ij} that are not determined by Eqs. (3.8) may be determined by the initial conditions and in this process b_{00} may be fixed too. In the exponential function in Eq. (3.7) there are no negative coefficients and because of this when $t \to \infty$ the system comes to the stationary solution from Eqs. (3.6). The form of this stationary solution is

$$\begin{aligned}x_i^* &= \frac{\prod_{j=1}^{i}[\alpha_{i-j} + (i-j)\beta_{i-j}]}{\prod_{j=1}^{i}(\alpha_j + j\beta_j + \gamma_j)} x_0^*,\ i = 1, \dots, N-1\\ x_N^* &= \frac{\prod_{j=1}^{N}[\alpha_{N-j} + (N-j)\beta_{N-j}]}{\gamma_N \prod_{j=1}^{N-1}(\alpha_j + j\beta_j + \gamma_j)} x_0^*\end{aligned} \qquad (3.9)$$

Let $x^* = \sum_{i=0}^{N} x_i$ be the total amount of the substance for the case of stationary state of the channel. Then we can consider the distribution $y_i^* = x_i^*/x^*$. Its form is

$$\begin{aligned}y_i^* &= \frac{\prod_{j=1}^{i}[\alpha_{i-j} + (i-j)\beta_{i-j}]}{\prod_{j=1}^{i}(\alpha_j + j\beta_j + \gamma_j)} y_0^*,\ i = 1, \dots, N-1\\ y_N^* &= \frac{\prod_{j=1}^{N}[\alpha_{N-j} + (N-j)\beta_{N-j}]}{\gamma_N \prod_{j=1}^{N-1}(\alpha_j + j\beta_j + \gamma_j)} y_0^*\end{aligned} \qquad (3.10)$$

To the best of our knowledge the distribution presented by Eq. (3.10) (Figs. 6–8) was not discussed by other authors. Let us consider the particular case where $\alpha_i = \alpha$, $\beta_i = \beta$ and $\gamma_i = \gamma$ and $k = \alpha/\beta$, $a = \gamma/\beta$. In this case the distribution from Eq. (3.10) is reduced to the distribution:

$$\begin{aligned}P(\zeta = i) &= P(\zeta = 0)\frac{(k-1)^{[i]}}{(a+k)^{[i]}};\ \ k^{[i]} = \frac{(k+i)!}{k!};\ i = 0, \dots, N-1\\ P(\zeta = N) &= \frac{P(\zeta = 0)}{a}\,\frac{(k-1)^{[N]}}{(a+k)^{[N-1]}},\end{aligned} \qquad (3.11)$$

$P(\zeta = 0) = y_0^* = x_0^*/x^*$ is the percentage of substance that is located in the first node of the channel. Let this percentage be

$$y_0^* = \frac{a}{a+k} \qquad (3.12)$$

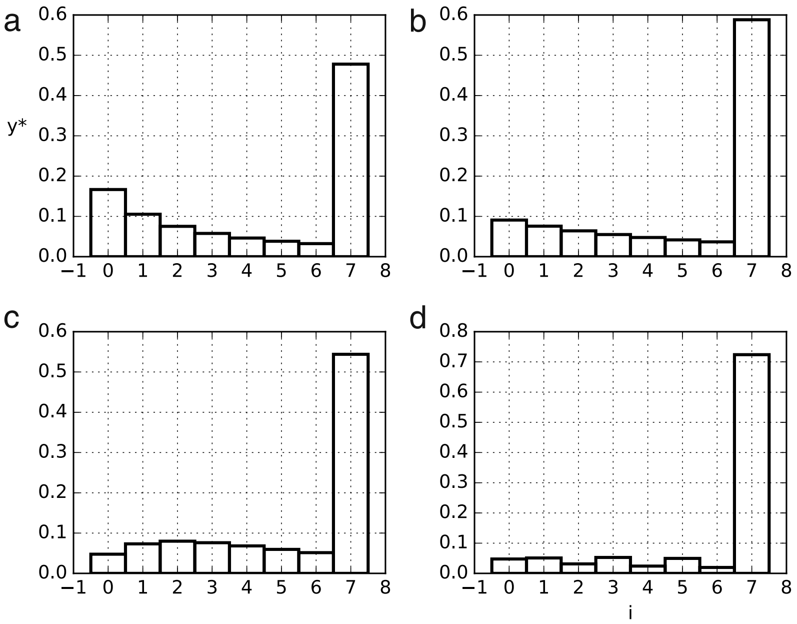

Fig. 6. The distribution (3.10). The channel consists of 8 nodes. Fixed parameters of the distribution are $\beta_i = 0.01$, $\gamma_i = 0.01$, $i = 0, \ldots, 7$. The influence of parameters α_i on the shape of distribution is shown as follows. (a) $\alpha_i = 0.05 + 0.1i/(i + 10)$. The effect of concentration of substance in the last node of the finite channel is clearly visible. (b) $\alpha_i = 0.1$. All parameters of the distribution do not depend on *i*. This may lead to even larger probability connected to the last node in comparison to some cases where α_i increases with increase of *i* (compare (a) and (b)). (c) $\alpha_i = 0.2 - 0.18i/(i + 1)$. The decrease of α_i with increasing *i* can lead to significant changes of the shape of distribution (3.10). (d) $\alpha_i = 0.2 + 0.2(-1)^i i/(i + 5)$. Large differences of α_i for neighboring values of *i* can influence the shape of the distribution. Note that the effect of large probability connected with the last node is presented in all the figures (a)–(d).

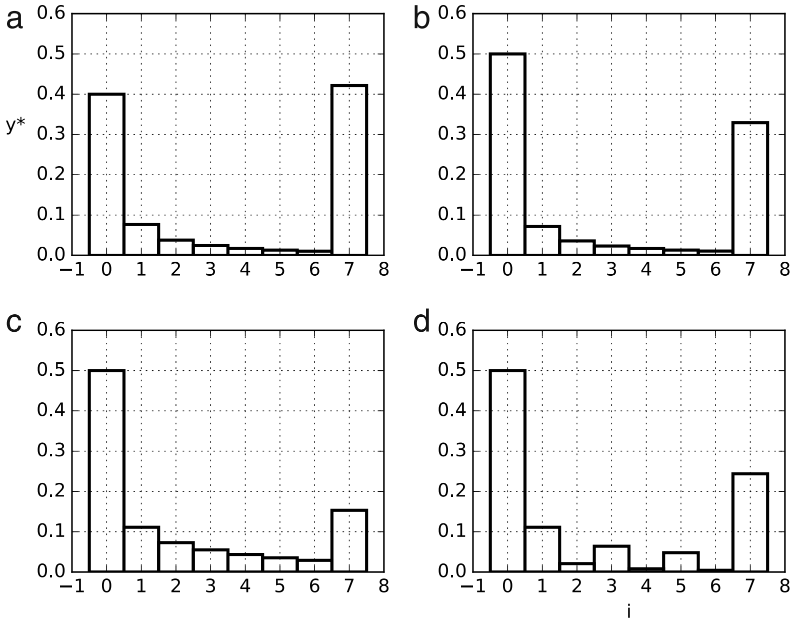

Fig. 7. The distribution (3.10) for a channel that consists of 8 nodes. Fixed parameters of the distribution are $\alpha_i = 0.01$, $\gamma_i = 0.01$, $i = 0, \ldots, 7$. The influence of parameters β_i on the shape of distribution is shown as follows. (a) $\beta_i = 0.05 + 0.04i/(i + 10)$. The largest probabilities are connected to the entry node of the channel and to the last node of the channel. The effect of concentration of substance in the last node of the channel is also presented. (b) $\beta_i = 0.05$. All parameters of the distribution do not depend on *i*. This may lead to larger probability connected to the first node of the channel in comparison to probability connected to last node of the channel. (c) $\beta_i = 0.05 - 0.05i/(i + 1)$. The decrease of β_i with increasing *i* can lead to further increase of probability connected to the entry node of the channel at the expense of the probability connected to the last node of the channel. (d) $\beta_i = 0.05 + 0.05(-1)^i i/(i + 1)$. Large differences of the values of β_i for neighboring values of *i* can influence the shape of the distribution and can lead to increase of the value of probability connected to the last node of the channel.

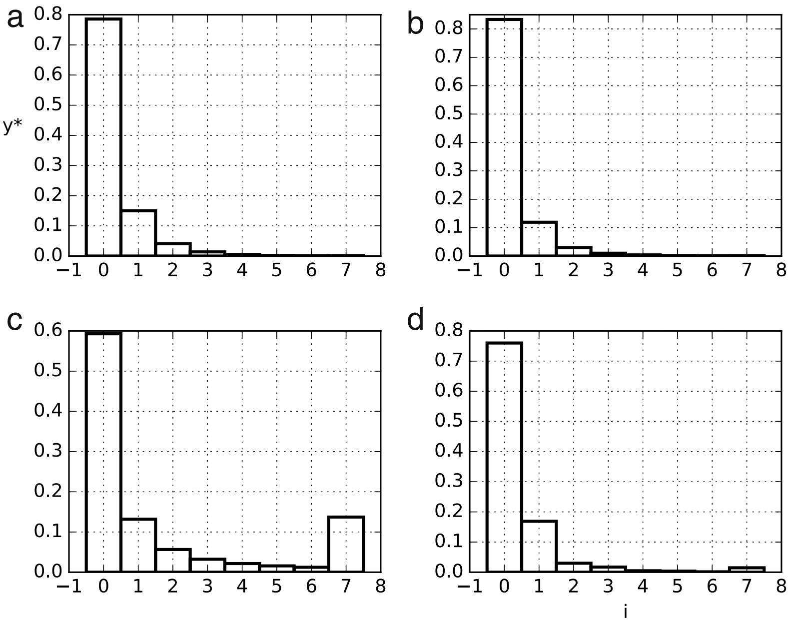

Fig. 8. The distribution (3.10) for a channel that has 8 nodes. Fixed parameters of the distribution are $\alpha_i = 0.01$, $\beta_i = 0.01$, $i = 0, \dots, 7$. The influence of parameters γ_i on the shape of distribution is shown as follows. (a) $\beta_i = 0.05 + 0.04i/(i + 10)$. The largest probabilities are connected to the entry node of the channel. Increasing value of γ_i with increasing i can lead to a fast decrease of values of probability for the last nodes of the channel. This can influence in a significant way the concentration of substance in the last node of the channel. (b) $\gamma_i = 0.05$. All parameters of the distribution do not depend on i. The probability connected to the entry node of the channel is still very large and the probabilities connected to the tail of the distribution are small. (c) $\gamma_i = 0.05 - 0.05i/(i + 1)$. The decrease of β_i with increasing i can lead to decrease of the value of probability connected to the entry node of the channel and to strengthening of the effect of concentration of substance in the last node of the channel. (d) $\gamma_i = 0.05 + 0.05(-1)^i i/(i + 1)$. Large differences of the values of β_i for neighboring values of i can influence the shape of the distribution and can lead to decrease of the value of probability connected to the last node of the channel.

The case described by Eq. (3.12) corresponds to the situation where the amount of substance in the first node is proportional of the amount of substance in the entire channel (self-reproduction property of the substance). In this case Eq. (3.11) is reduced to the truncated Waring distribution (Fig. 9):

$$P(\zeta = i) = \frac{a}{a+k} \frac{(k-1)^{[i]}}{(a+k)^{[i]}}; \quad k^{[i]} = \frac{(k+i)!}{k!}; \ i = 0, \dots, N-1$$
$$P(\zeta = N) = \frac{1}{a+k} \frac{(k-1)^{[N]}}{(a+k)^{[N-1]}}, \tag{3.13}$$

The truncated Waring distribution (3.13) is close to the Waring distribution that was discussed above in the text. A characteristic feature of the truncated Waring distribution is the possibility for accumulation of substance in the last node of the channel (and this concentration can be quite significant) [47].

Let us note that for the case of distribution (3.10) $y^*_N > y^*_{N-1}$ when $k+(N-1) > a_N$, i.e., the concentration of the substrate in the last node of the channel depends on the situation in the last two nodes (from the parameters $a_{N-1}, \beta_{N-1}, \gamma_N$). For the case of truncated Waring distribution (3.13) $y^*_N > y^*_{N-1}$ when $k + (N - 1) > a$ where $a = \sigma/\beta$, i.e., the concentration of substance in the last node of the channel depends on the situation in the first node of the channel. This may be important for the case of channels for human migrants.

3.2. Non-stationary regime of functioning of the channel

In the nonstationary case $dx_0/dt \neq 0$. Below we shall consider the case $dx_0/dt > 0$. In this case the solution of the first equation of the system of Eqs. (3.5) is

$$x_0 = b_{00} \exp[(\sigma_0 - \alpha_0 - \gamma_0)t] \tag{3.14}$$

where b_{00} is a constant of integration. x_i must be obtained by solution of the corresponding Eqs. (3.5). The form of x_i is

$$x_i = \sum_{j=0}^{i} b_{ij} \exp[-(\alpha_j + j\beta_j + \gamma_j - \sigma_j)t] \tag{3.15}$$

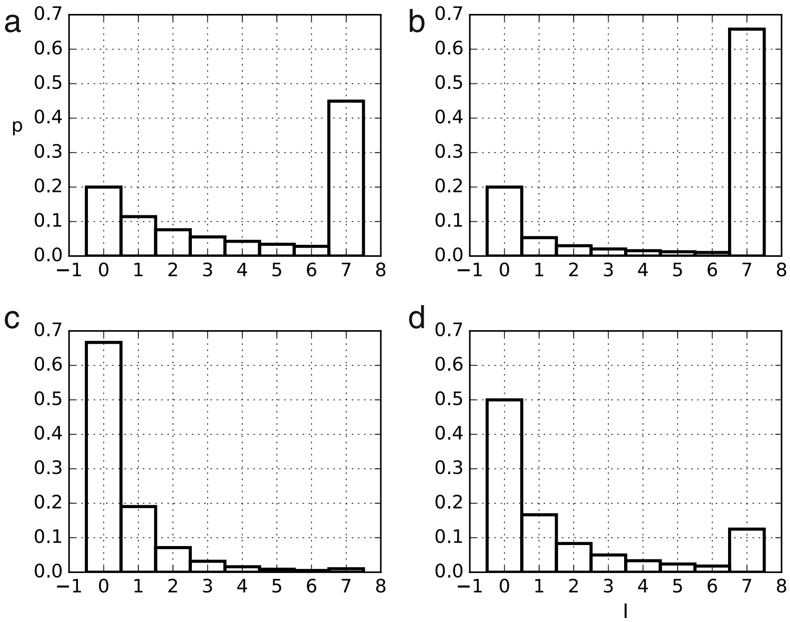

Fig. 9. Truncated Waring distribution (3.13) for a channel consisting of 8 nodes. (a) $k = 2$, $a = 0.5$. The effect of a large probability connected to the last node of the channel is clearly visible. (b) $k = 0.4$, $a = 0.1$. There is no change in the shape of distribution. The probability connected to the last node of the channel increases at the expense of probabilities connected to the intermediate nodes of the channel. (c) $k = 2$, $a = 5$. The value of the parameter k is the same as in (a). The comparison of distributions from (a) and (c) shows the effect of increasing values of parameter a: There is an increase of the probability connected to the entry node of the channel at the expense of the probabilities of the nodes from the second half of the channel that are connected to the tail of the distribution. (d): $k = 5$, $a = 0.5$. The value of the parameter a is the same as in (a). Thus by comparing figures (a) and (d) one can observe the effect of increasing value of parameter k on the shape of the truncated Waring distribution. There is an increase of the probability connected to the first two nodes of the channel at the expense of the probability connected to the last node of the channel. We note that the truncated Waring distribution depends on two parameters and the more general distribution (3.10) depends on 3 parameters.

In order to understand the processes in the channel let us consider first the case of channel consisting of two nodes ($N = 1$). In this case we have to solve the additional equation

$$\frac{dx_1}{dt} = \alpha_0 x_0 - \gamma_1 x_1 \tag{3.16}$$

The solution is

$$x_1 = \frac{\alpha_0}{\gamma_1 - \gamma_0 + \sigma_0 - \alpha_0} b_{00} \exp[(\sigma_0 - \alpha_0 - \gamma_0)t] + b_{11} \exp(-\gamma_1 t) \tag{3.17}$$

b_{11} can be determined from the initial conditions at $t = 0$. The asymptotic form of the obtained solution ($t \to \infty$) is

$$\begin{aligned} x_0^a &= b_{00} \exp[(\sigma_0 - \alpha_0 - \gamma_0)t] \\ x_1^a &= \frac{\alpha_0}{\gamma_1 - \gamma_0 + \sigma_0 - \alpha_0} b_{00} \exp[(\sigma_0 - \alpha_0 - \gamma_0)t] \end{aligned} \tag{3.18}$$

as $\gamma_1 > 0$. Let us consider the asymptotic distribution $y_i^a = x_i^a / x^a$ where $x^a = \sum_{i=0}^{N} x_i$. x_0^a and x_1^a depend on t but nevertheless the asymptotic distribution is stationary

$$y_0^a = \frac{1}{1 + \frac{\alpha_0}{\gamma_1 - \gamma_0 + \sigma_0 - \alpha_0}}; \quad y_1^a = \frac{1}{1 + \frac{\gamma_1 - \gamma_0 + \sigma_0 - \alpha_0}{\alpha_0}} \tag{3.19}$$

Let us now consider the case of channel containing more than 2 nodes ($N > 1$). In this case the solution of the system of Eqs. (3.5) is (3.15) where $\sigma_i = 0, i = 1, \dots, N-1$;

$$\begin{aligned} b_{ij} &= \frac{\alpha_{i-1} + (i-1)\beta_{i-1}}{(\alpha_i - \alpha_j) + (i\beta_i - j\beta_j) + (\gamma_i - \gamma_j)} b_{i-1,j}; \; i = 1, \dots, N-1 \\ b_{Nj} &= \frac{\alpha_{N-1} + (N-1)\beta_{N-1}}{\gamma_N - \gamma_j - \alpha_j - j\beta_j} b_{N-1,j}, \; j = 0, \dots, N-1 \\ \sigma_N &= \alpha_N + N\beta_N, \end{aligned} \tag{3.20}$$

and b_{ii} are determined from the initial conditions in the nodes of the channel. The asymptotic solution ($t \to \infty$) is

$$x_i^a = b_{i0} \exp[(\sigma_0 - \alpha_0 - \gamma_0)t] \tag{3.21}$$

This means that the asymptotic distribution $y_i^a = x_i^a / x^a$ is stationary

$$y_i^a = \frac{b_{i0}}{\sum_{j=0}^{N} b_{j0}} \tag{3.22}$$

regardless of the fact that the amount of substance in the two cells many increase or decrease exponentially. The explicit form of this distribution is

$$\begin{aligned}
y_0^a &= \frac{1}{\Omega} \\
y_i^a &= \frac{\prod_{k=1}^{i} \frac{\alpha_{i-k}+(i-k)\beta_{i-k}}{(-\alpha_0+\alpha_{i-k+1})+(i-k+1)\beta_{i-k+1}+(-\gamma_0+\gamma_{i-k+1})}}{\Omega}, \\
i &= 1, \dots, N-1 \\
y_N^a &= \frac{\frac{\alpha_{N-1}+(N-1)\beta_{N-1}}{(\gamma_N-\gamma_0)-\alpha_0} \prod_{k=1}^{N-1} \frac{\alpha_{i-k}+(i-k)\beta_{i-k}}{(-\alpha_0+\alpha_{i-k+1})+(i-k+1)\beta_{i-k+1}+(-\gamma_0+\gamma_{i-k+1})}}{\Omega}
\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}
\Omega &= 1 + \sum_{i=1}^{N-1} \prod_{k=1}^{i} \frac{\alpha_{i-k} + (i-k)\beta_{i-k}}{(-\alpha_0 + \alpha_{i-k+1}) + (i-k+1)\beta_{i-k+1} + (-\gamma_0 + \gamma_{i-k+1})} + \\
&\quad \frac{\alpha_{N-1} + (N-1)\beta_{N-1}}{(\gamma_N - \gamma_0) - \alpha_0} \prod_{k=1}^{N-1} \frac{\alpha_{i-k} + (i-k)\beta_{i-k}}{(-\alpha_0 + \alpha_{i-k+1}) + (i-k+1)\beta_{i-k+1} + (-\gamma_0 + \gamma_{i-k+1})}
\end{aligned} \tag{3.24}$$

4. Application of the model to channels of migration networks

4.1. Main outcome of the model for the case of human migration through a migration channel consisting of chain of countries. Meaning of the model parameters

The model discussed above can be used for a study of motion of substance through cells of appropriate technological systems. The model can be applied also for investigation of other systems. Below we shall discuss it in connection with channels of human migration for the case when the migration flows are large and continuous approximation of these flows can be used. The main outcome of the model for the case of human migration is the distribution (3.10) that describes the distribution of migrants in the countries of the channel for the case of stationary regime of functioning of the migration channel. Below we shall consider several additional outcomes of the model.

Let us consider a chain of $N + 1$ countries or cities. This chain may be considered as a channel in a migration network. The nodes of this network (corresponding to the countries of the channel for an example) may be considered as boxes (cells). A flow of migrants moves through this migration channel from the country of entrance to the final destination country. The incoming country will be the node with label 0 and the final destination country will be the node with label N. Let us have a number x of migrants that are distributed among the countries. Let x_i be the number of migrants in the ith country. This number can change on the basis of the following three processes: (i) A number s of migrants enter the channel from the external environment through the 0th node (country of entrance); Amount f_i from x_i is transferred from the ith country to the $i + 1$th country; (iii) Amount g_i from x_i change their status (e.g. they are not anymore migrants and may become citizens of the corresponding country, may return home, etc.). The values of x_i can be determined by Eqs. (3.3). The relationships (2.6) mean that: (i) The number of migrants s that enter the channel is proportional of the current number of migrants in the incoming country of the channel; (ii) There may be preference for some countries, e.g. migrants may prefer the countries that are around the end of the migration channel (and the final destination country may be the most preferred one); (iii) It is assumed that the conditions along the channel are different with respect to 'leakage' of migrants, e.g. the different rates γ_i of migrants leave the flow of migrants in different countries of the channel. In addition the transition from country to country may have different grade of difficulty (different α_i) and the attractiveness of the countries along the channel may be different for migrants (different β_i).

σ_0 is the "gate" parameter as it regulates the number of migrants that enter the channel. The parameters γ_i regulate the "absorption" of the channel as they reflect the change of the status of some migrants. The large values of γ_i may compensate for the value of σ and even may lead to decrease of the number of migrants in the channel. The large values of γ_i may however lead to integration problems connected to migrants.

Small values of parameters α_i mean that the way of the migrants through the channel is more difficult and because of this the migrants tend to concentrate in the entry country (and eventually in the second country of the channel). The countries that are in the second half of the migration channel and especially the final destination country may try to decrease α_i by agreements that commit the entry country to keep the migrants on its territory. Any increase of α_i may lead to increase of the proportion of migrants that reach the second half of the migration channel and especially the final destination country.

The parameters β_i regulate the attractiveness of the countries along the migration channel. Large values of β_i mean that the remaining countries in the channel and especially the final destination country are very attractive for some reason. If for some reason β_i are kept at high values a flood of migrants may reach the final destination country which may lead to large logistic and other problems.

4.2. Concentration of migrants in a selected country of the migration channel

Let us now consider the case of channel consisting of finite number of countries and the stationary case of the functioning of the channel. Then from Eq. (3.6) we obtain the relationship

$$\frac{y_N^*}{y_{N-1}^*} = \frac{\alpha_{N-1} + (N-1)\beta_{N-1}}{\gamma_N} \tag{4.1}$$

If $\frac{y_N^*}{y_{N-1}^*} > 1$ there is an effect of concentration of migrants in the final destination country. This happens when

$$\alpha_{N-1} + (N-1)\beta_{N-1} > \gamma_N \tag{4.2}$$

i.e., if (i) the attractiveness of the final destination country is large (large value of β_{N-1}); (ii) it is relatively easy to cross the border to the final destination country (large α_{N-1}) and (iii) the probability of change of the status of migrants in the final destination country of the channel are not large enough to compensate for the popularity of the final destination country (value of γ_N is relatively small). In order to avoid the arising of the effect of the concentration of migrants in the final destination country one has to achieve

$$\alpha_{N-1} + (N-1)\beta_{N-1} < \gamma_N \tag{4.3}$$

This means that one should try to decrease the number of migrants entering the final destination country (to lower the value of α_{N-1}); to decrease the attractiveness of the final destination country (to lower the value of β_{N-1}) and to increase the probabilities for change of the status of the migrants in the final destination country (to increase the value of γ_N).

A new effect with respect to the theory developed in our previous study [47] is the possibility of accumulation of migrants not only in the final destination country but also in any country of the channel. This can happen (see Eqs. (3.6)) when (for some value of i)

$$\frac{\alpha_{i-1} + (i-1)\beta_{i-1}}{\alpha_i + i\beta_i + \gamma_i} > 1 \tag{4.4}$$

Eq. (4.4) shows that such a case of concentration of migrants may happen when the entry of migrants in the ith country is easy and this country is popular among migrants (large values of α_{i-1} and β_{i-1} with respect to α_i and β_i) and in addition the value of γ_i is small.

4.3. Total number of migrants in the countries of the migration channel

The change of the total number of migrants in the channel can be obtained by taking the sum of the Eqs. (3.5). The result is

$$\frac{dx}{dt} = \sigma_0 x_0 - \sum_{l=0}^{N} \gamma_l x_l \tag{4.5}$$

Thus the total number of migrants in the channel may increase fast when many migrants enter the channel (when the value of σ_0 is large) and decrease when the probability for change of the status of the migrants is large (when the values of some of γ_l or the values of all γ_l are large).

In the stationary regime of functioning of a finite channel the total number of migrants in the countries of the channel is

$$x^* = x_0^* \left\{ 1 + \sum_{i=1}^{N-1} \frac{\prod_{j=1}^{i}[\alpha_{i-j} + (i-j)\beta_{j-1}]}{\prod_{j=1}^{i}(\alpha_j + j\beta_j + \gamma_j)} + \frac{\prod_{j=1}^{N}[\alpha_{i-j} + (N-j)\beta_{j-1}]}{\gamma_N \prod_{j=1}^{N-1}(\alpha_j + j\beta_j + \gamma_j)} \right\} \tag{4.6}$$

Thus the number of migrants in the countries of the channel may be decreased if one manages to decrease the number of migrants in the entry country of the channel (i.e. if one manages to decrease the value of x_0^*).

4.4. Case of small permeability of the borders between the countries of the channel and large attractiveness of the final destination country. Appearance of Zipf distribution

We consider the case of channel consisting of finite number of countries and in stationary regime of functioning. Let in addition the motion of migrants be determined by the attractiveness of the final destination country at the expense of the easiness of moving through the borders between the countries of the channel (i.e., $\beta_i \gg \alpha_i$ and parameter α_i may be neglected in the relationships for all countries except in the numerator for the 0th country in Eq. (3.6)). Let in addition there be no "leakage" (the migrants are very much fascinated by the final destination country and the change of the statuses along the channel is small, i.e., one can neglect all γ_i except for γ_N which is assumed to be significant). Thus from Eq. (3.6) we obtain

$$x_1^* = \frac{\alpha_0}{\beta_1} x_0^*;\ x_i^* = \frac{(i-1)}{i}\frac{\beta_{i-1}}{\beta_i} x_{i-1}^*,\ i = 2, \dots, N-1;\ x_N^* = \frac{(N-1)\beta_{N-1}}{\gamma_N} x_{N-1}^*. \tag{4.7}$$

From the second of Eqs. (4.7) one easily obtains $x_i^* = \frac{i-k}{i}\frac{\beta_{i-k}}{\beta_i} x_{i-k}^*$. Then the approximate total number of the migrants in the channel is

$$x^* = x_0^* \left\{ 1 + \alpha_0 \sum_{l=1}^{N-1} \frac{1}{l\beta_l} + \frac{\alpha_0}{\gamma_N} \right\} \tag{4.8}$$

and the distribution of the migrants among the countries of the channel is

$$\begin{aligned}
y_0^* &= \frac{1}{1 + \alpha_0 \sum_{l=1}^{N-1} \frac{1}{l\beta_l} + \frac{\alpha_0}{\gamma_N}} \\
y_1 &= \frac{\alpha_0}{\beta_1 \left[1 + \alpha_0 \sum_{l=1}^{N-1} \frac{1}{l\beta_l} + \frac{\alpha_0}{\gamma_N} \right]} \\
y_i^* &= \frac{\alpha_0}{i\beta_i \left[1 + \alpha_0 \sum_{l=1}^{N-1} \frac{1}{l\beta_l} + \frac{\alpha_0}{\gamma_N} \right]}, \quad i = 2, \dots, N-1 \\
y_N^* &= \frac{\alpha_0}{\gamma_N \left[1 + \alpha_0 \sum_{l=1}^{N-1} \frac{1}{l\beta_l} + \frac{\alpha_0}{\gamma_N} \right]}
\end{aligned} \tag{4.9}$$

Let us now denote α_0 as α and assume that $\beta_1 = \dots = \beta_{N-1} = \beta$. Then

$$x^* = x_0^* \left\{ 1 + \frac{\alpha}{\beta} \sum_{l=1}^{N-1} \frac{1}{l} + \frac{\alpha}{\gamma_N} \right\} = x_0^* \left\{ 1 + \frac{\alpha}{\beta} H_{N-1} + \frac{\alpha}{\gamma} \right\} \tag{4.10}$$

where H_{N-1} is the $N-1$th harmonic number. Let us use the approximate relationship for harmonic numbers $H_N = \ln(N) + C_E + \frac{1}{2N} - \frac{1}{12N^2} + \frac{1}{120N^4} - \cdots$ (C_E is the constant of Euler). Then for a migration channel of finite and not very large length we obtain the relationship

$$x^* \approx x_0^* \left\{ 1 + \frac{\alpha}{\beta} \left[\ln(N-1) + C_E + \frac{1}{2(N-1)} \right] + \frac{\alpha}{\gamma} \right\} \tag{4.11}$$

and the approximate distribution of the migrants in the channel $y_i^* = \frac{x_i^*}{x^*}$ will be

$$\begin{aligned}
y_0^* &= \frac{1}{\left\{ 1 + \frac{\alpha}{\beta} \left[\ln(N-1) + C_E + \frac{1}{2(N-1)} \right] + \frac{\alpha}{\gamma} \right\}} \\
y_1^* &= \frac{\alpha}{\beta \left\{ 1 + \frac{\alpha}{\beta} \left[\ln(N-1) + C_E + \frac{1}{2(N-1)} \right] + \frac{\alpha}{\gamma} \right\}} \\
y_i^* &= \frac{\alpha}{i\beta \left\{ 1 + \frac{\alpha}{\beta} \left[\ln(N-1) + C_E + \frac{1}{2(N-1)} \right] + \frac{\alpha}{\gamma} \right\}}, i = 2, \dots, N-1 \\
y_N^* &= \frac{\alpha}{\gamma_N \left\{ 1 + \frac{\alpha}{\beta} \left[\ln(N-1) + C_E + \frac{1}{2(N-1)} \right] + \frac{\alpha}{\gamma} \right\}}
\end{aligned} \tag{4.12}$$

which is a version of truncated Zipf distribution.

4.5. Two scenarios for a migration channel consisting of 3 countries. what is more effective if the number of migrants has to be decreased: (i) to decrease permeability of the borders between the countries of the channel or (ii) to decrease the influx of migrants into the incoming country of the channel?

For an additional illustration of the above theory we shall discuss a stationary regime of functioning of a migration channel, containing 3 countries. Let the migrants come overseas (e.g., by boats) to the incoming country of the channel. The attractive country in the channel is the third country of the channel (the final destination country). The second country of the channel is not attractive for the migrants but in order to reach the final destination country the migrants have to move through its territory. Let the flow of migrants be large. Then we can apply the part of the model described by Eqs. (3.9) and (3.10). The numbers of the migrants in the three countries of the channel are

$$x_0^*; \quad x_1^* = x_0^* \frac{\alpha_0}{\alpha_1+\beta_1+\gamma_1}; \quad x_2^* = x_0^* \frac{\alpha_1+\beta_1}{\gamma_2}\frac{\alpha_0}{\alpha_1+\beta_1+\gamma_1} \tag{4.13}$$

The total number of migrants in the countries of the channel will be

$$x^* = x_0^* \left(1 + \frac{\alpha_0}{\alpha_1+\beta_1+\gamma_1} + \frac{\alpha_1+\beta_1}{\gamma_2}\frac{\alpha_0}{\alpha_1+\beta_1+\gamma_1}\right) \tag{4.14}$$

and the distribution of the migrants along the countries of the channel will be

$$\begin{aligned} y_0^* &= \frac{1}{\left(1+\frac{\alpha_0}{\alpha_1+\beta_1+\gamma_1}+\frac{\alpha_1+\beta_1}{\gamma_2}\frac{\alpha_0}{\alpha_1+\beta_1+\gamma_1}\right)} \\ y_1^* &= \frac{\frac{\alpha_0}{\alpha_1+\beta_1+\gamma_1}}{\left(1+\frac{\alpha_0}{\alpha_1+\beta_1+\gamma_1}+\frac{\alpha_1+\beta_1}{\gamma_2}\frac{\alpha_0}{\alpha_1+\beta_1+\gamma_1}\right)} \\ y_2^* &= \frac{\frac{\alpha_1+\beta_1}{\gamma_2}\frac{\alpha_0}{\alpha_1+\beta_1+\gamma_1}}{\left(1+\frac{\alpha_0}{\alpha_1+\beta_1+\gamma_1}+\frac{\alpha_1+\beta_1}{\gamma_2}\frac{\alpha_0}{\alpha_1+\beta_1+\gamma_1}\right)} \end{aligned} \tag{4.15}$$

Let us discuss two scenarios. In the first scenario there are no measures to decrease the number of migrants in the incoming country of the channel and their number remains, e.g., $x_0^* = 300{,}000$. Let the stationary state of the channel be characterized by values of parameters: $\alpha_0 = \alpha_1 = \beta_1 = \gamma_1 = \gamma_2 = 0.0001$. Then in countries of the channel there are 600,000 migrants: 300,000 in the incoming country (50%), 100,000 in the second country of the channel (about 16, 7%) and 200,000 in the final destination country (about 33.3%). Now let the number of the migrants in the first country of the channel be a large burden for this country, and it decides to increase α_0 (e.g., to ease the border control and to increase the probability that the migrants may move successfully (and mostly illegally) to the second country of the channel). Let the result of such a behavior be that α_0 increases from 0.0001 to 0.0002. After some time the channel will have new stationary state of operation where the number of migrants in the incoming country of the channel will be 300,000 (x_0^* remains unchanged) but the number of migrants in the second and third country of the channel will increase to 200,000 in the second country of the channel and 400,000 in the final destination country. Then the second country of the channel may decide that the migration burden is too high for it and this country may take measures that may lead to increase of α_1. Let this increase be from $\alpha_1 = 0.0001$ to $\alpha_1 = 0.0002$. We remember that $\alpha_0 = 0.0002$ and the other parameters of the channel have values of 0.0001. This will have the following effect of the stationary regime of the functioning of the channel: there will be 300,000 migrants in the incoming country of the channel (as no measures are taken to decrease x_0^*). Then there will be 150,000 migrants in the second country of the channel (increasing value of α_1 decreased the number of migrants in this country) and there will be 450,000 migrants in the final destination country (part of the migrants move from the second country of the channel to the final destination country). Let now the final destination country decides that the migration burden is too large for it and this country takes measures to decrease its popularity. Let these measures lead to a new value of β_1: $\beta_1 = 0$. Then the stationary regime of the functioning of the migration channel will be characterized by the following number of migrants in the three countries: 300,000 migrants in the incoming country, 200,000 migrants in the second country of the channel, and 400,000 migrants in the final destination country of the channel. Thus the number of migrants in the final destination country will decrease at the expense of the migrants in the second country of the channel.

The sole actions of the countries from the scenario 1 above can continue but as we have seen this will not lead to significant decrease of migration. Such significant decrease can be realized in the scenario 2: the countries concentrate their efforts in order to decrease x_0^* keeping the values of the other parameters unchanged. This will have large effect as follows. Let the initial stationary state of the channel be characterized by $x_0^* = 300{,}000$, and $\alpha_0 = \alpha_1 = \beta_1 = \gamma_1 = \gamma_2 = 0.0001$. Then the number of migrants in the second country of the channel will be 100,000 and the number of migrants in the final destination country will be 200,000. Let now the measures be taken and the number of migrants in the incoming country of the channel decreases to $x_0^* = 200{,}000$. Then the stationary state of the channel will be characterized by about 67,000 migrants in the second country of the channel and about 133,000 migrants in the final destination country. If the measures for decreasing x_0^* continue and the number of migrants reduces to 45,000 then the corresponding numbers of migrants will be 15,000 in the second country of the channel and 30,000 in the final destination country. Thus scenario 2 is much more effective from the point of view of decreasing migration in the countries of the channel.

5. Concluding remarks

Above we have discussed a model for motion of a substance in a channel consisting of nodes of a network. Two regimes of functioning of the channel are studied: stationary regime of motion of the substance and nonstationary regime of motion of the substance. The main outcome of the study are the obtained distributions of substance along the cells for the case of stationary regime of motion of the substance. The corresponding distribution for the case of channel containing infinite number of modes is a distribution with a very long tail that is a generalization of the Waring distribution. And the Waring distribution contains as particular cases the famous Zipf distribution as well as the Yule–Simon distribution. The distribution obtained for the case of a channel containing finite number of nodes is a generalization of the truncated Waring distribution that was discussed in [47]. In addition to classical application of the model for calculation of motion and distribution of substance in channels of technological systems we have discussed the model also in the context of motion of large numbers of migrants through migration channels, e.g., connecting several countries. Specific characteristic of the discussed model is the possibility for different "leakage" of the nodes, i.e., the different probabilities for a change of the status of a migrant in the different nodes (countries) of the channel. Several outcomes of the model for the case of migration are as follows

1. For the case of non-stationary regime of functioning of the channel the number of migrants in the channel may increase or decrease exponentially but the asymptotic distribution of the migrants in the countries of the channel is stationary and depends strongly on the situation in the first node (incoming country) of the channel. The corresponding distributions of the migrants in the countries of the channel are more complicated in comparison to the distributions for the case of stationary regime of functioning of the channel.
2. In the stationary regime of functioning of a channel consisting of finite number of nodes an effect of concentration of migrants in the final destination country (last node of the channel) may be observed if the final destination country is popular enough.
3. The possibility of different "leakage" and different preferences may lead to another concentration effect: the concentration of migrants may happen not only in the last country of the channel but also in the countries that are between the incoming country of the channel and the final destination country.
4. If the popularity of the countries close to the final destination country is large and the motion from node to node (from country to country) is not so easy and in addition the if migrants are not interested in change of their status in the countries of the channel except for the final destination country, then the distribution of the migrants in the countries of the channel is a version of the Zipf distribution.
5. Let the number of migrants in the countries of the channel has to be decreased and one has to choose between two possibilities: (i) to reduce the number of migrants arriving in the incoming country of the channel or (ii) to reduce permeability of the borders between the countries of the channel. The more effective way to deal with the problem is to realize the possibility (i).

Appendix. Proof that Eq. (2.8) is solution of Eqs. (2.7)

Let us consider the first equation from Eqs. (2.7). In this case $i = 0$ and Eq. (2.8) becomes $x_0 = x_0^* + b_{00} \exp[-(\alpha_0 + \gamma_0)t]$. The substitution of the last relationship in the first of the Eqs. (2.7) and accounting for Eq. (2.9) leads to the relationship

$$0 = (\sigma_0 - \alpha_0 - \gamma_0)x^* + b_{00}\sigma_0 \exp[-(\alpha_0 + \gamma_0)t] \qquad (A.1)$$

Let us assume $\sigma_0 = \alpha_0 + \gamma_0$ and $b_{00} = 0$. Then Eq. (2.8) describes the solution of the first of Eqs. (2.7).

Let us now consider Eqs. (2.7) for $i = 1, 2, \dots$. Let us fix i and substitute Eq. (2.8) in the corresponding equation from Eqs. (2.7). The result is

$$\sum_{j=0}^{i-1} \exp[-(\alpha_j + jb_j + \gamma_j)t]\{-b_{ij}(\alpha_j + j\beta_j + \gamma_j) - b_{i-1,j}[\alpha_{i-1} + (i-1)\beta_{i-1}] + b_{ij}(\alpha_i + i\beta_i + \gamma_i)\} = 0 \qquad (A.2)$$

As it can be seen from Eq. (2.10) Eq. (A.2) is satisfied. Thus we have shown that Eq. (2.8) is solution of Eqs. (2.7).

References

[1] L.D. Ford Jr, D.R. Fulkerson, Flows in Networks, Princeton University Press, Princeton, NJ, 1962.
[2] R.K. Ahuja, T.L. Magnanti, J.B. Orlin, Network flows, in: Theory, Algorithms, and Applications, Prentice Hall, NJ, 1993.
[3] W.-K. Chan, Theory of Nets: Flows in Networks, Wiley, New York, 1990.
[4] N.H. Gartner, G. Improta (Eds.), Urban traffic networks, in: Dynamic Flow Modeling and Control, Springer, Berlin, 1995.
[5] L. Ambrosio, A. Bressan, D. Helbing, A. Klar, E. Zuazua (Eds.), Modeling and Optimisation of Flows on Networks, Springer, Heidelberg, 2010.
[6] M. Rossvall, A.C. Esquivel, A. Lancichinetti, J.D. West, R. Lambiotte, Memory in network flows and its effects on spreading dynamics and community detection, Nature Commun. 5 (2014) Article No. 4630.
[7] E. Bernard, L. Jacob, J. Mairal, J.-P. Vert, Efficient RNA isoform identification and quantification from RNA-Seq data with network flows, Bioinformatics 30 (2014) 2447–2455.

[8] R.E. Gomori, T.C. Hu, Multi-terminal network flows, J. Soc. Ind. Appl. Math. 9 (1961) 551–570.
[9] D. Bertsimas, M. Sim, Robust discrete optimization and network flows, Math. Program. 98 (2003) 49–71.
[10] J.E. Aronson, A survey of dynamic network flows, Ann. Oper. Res. 20 (1989) 1–66.
[11] D. Helbing, L. Buzna, A. Johansson, T. Werner, Self-organized pedestrian crowd dynamics: Experiments, simulations, and design solutions, Transp. Sci. 39 (2005) 1–24.
[12] M. Skutella, An introduction to network flows over time, in: W. Cook, L. Lovasz, J. Vygen (Eds.), Research Trends in Combinatorial Optimization, Springer, Berlin, 2009, pp. 451–482.
[13] M. Treiber, A. Kesting, Traffic Flow Dynamics: Data, Models, and Simulation, Springer, Berlin, 2013.
[14] A.V. Bozhenyuk, E.M. Gerasimenko, J. Kacprzyk, I. Naumovich, Flows in Networks under Fuzzy Conditions, Springer International Publishing Switzerland, 2017.
[15] E.S. Lee, A theory of migration, Demography 3 (1966) 47–57.
[16] R. Armitage, Population projections for English local authority areas, Popul. Trends 43 (Spring) (1986) 31–40.
[17] I. Bracken, J.J. Bates, Analysis of gross migration profiles in England and Wales: some developments in classification, Environ. Plan. 15 (1983) 343–355.
[18] A.G. Champion, G. Bramley, A.S. Fotheringham, J. Macgill, P.H. Rees, A migration modelling system to support government decision-making, in: J. Stillwell, S. Geertman (Eds.), Planning Support Systems in Practice, Springer Verlag, Berlin, 2002, pp. 257–278.
[19] J.R. Harris, M.P. Todaro, Migration, unemployment and development: A two-sector analysis, Amer. Econ. Rev. 60 (1970) 126–142.
[20] J.H. Simon, The Economic Consequences of Migration, The University of Michigan Press, Ann Arbor, MI, 1999.
[21] R. Skeldon, Migration and Development: A Global Perspective, Routledge, London, 1992.
[22] D.S. Massey, J. Arango, G. Hugo, A. Kouaougi, A. Pellegrino, J. Edward Taylor, Theories of international migration: a review and appraisal, Popul. Dev. Rev. 19 (1993) 431–466.
[23] W.J. Ethier, International trade and labor migration, Amer. Econ. Rev. 75 (1985) 691–707.
[24] G.J. Borjas, Economic theory and international migration, Int. Migration Rev. 23 (1989) 457–485.
[25] F.J. Willekens, Probability models of migration: complete and incomplete data, SA J. Demography 7 (1999) 31–43.
[26] J. Ledent, Multistate life table: movement versus transition perspectives, Environ. Plan. 12 (1980) 533–562.
[27] F. Willekens, Models of migration observations and judgment, in: J. Raymer, F. Willekens (Eds.), International Migration in Europe: Data, Models and Estimates, Wiley, New York, 2008, pp. 117–147.
[28] H.-P. Blossfeld, G. Rohwer, Techniques of Event History Modeling: New Approaches To Casual Analysis, Lawrence Erlbaum, New Jersey, 2002.
[29] H.P. Blossfeld, K. Golsch, G. Rohwer (Eds.), Event History Analysis with Stata, Lawrence Erlbaum, New Jersey, 2007.
[30] D.S. Hachen, The competing risk model, Sociol. Methods Res. 17 (1988) 21–54.
[31] B. Singer, S. Spilerman, Mathematical representations of development. Theories, in: J.R. Nesselroade, P.B. Baltes (Eds.), Longitudinal Research in the Study of Behavior and Development, Academic Press, New York, 1979, pp. 155–177.
[32] L. Collins, Industrial Migration in Ontario: Forecasting Aspects of Industrial Activity Through Markov Chains, Statistics Canada, Ottawa, 1972.
[33] L. Collins, An Introduction To Markov Chain Analysis, Headey Brothers Ltd, London, 1975.
[34] G.J. McLachlan, T. Krishnan, The EM Algorithm and Extensions, Wiley, New York, 1977.
[35] J. Raymer, The estimation of international migration flows: a general technique focused on the origin–destination association structure, Environ. Plan. 39 (2007) 985–995.
[36] M.J. Bierley, J.J. Forster, J.W. McDonald, P.W.F. Smith, Bayessian estimation of migration flows, in: J. Raymer, F. Willekens (Eds.), International Migration in Europe: Data, Models and Estimates, Wiley, New York, 2008, pp. 149–174.
[37] M.J. Greenwood, Modeling migration, in: K. Kemp-Leonard (Ed.), Encyclopedia of Social Measurement, Vol. 2, Elsevier, Amsterdam, 2005, pp. 725–734.
[38] J.T. Fawcet, Networks, linkages, and migration systems, Int. Migration Rev. 23 (1989) 671–680.
[39] D.T. Gurak, F. Caces, Migration networks and the shaping of migration systems, in: M.M. Kitz, L.L. Lim, H. Zlotnik (Eds.), International Migration Systems: A Global Approach, Clarendon Press, Oxford, 1992, pp. 150–176.
[40] N.K. Vitanov, Z.I. Dimitrova, M. Ausloos, Verhulst-Lotka–Volterra model of ideological struggle, Phys. A 389 (2010) 4970–4980.
[41] N.K. Vitanov, M. Ausloos, G. Rotundo, Discrete Model of Ideological Struggle Accounting for Migration, Advances in Complex Systems 15 (Suppl. 1) (2012) Article number 1250049.
[42] N.K. Vitanov, I.P. Jordanov, Z.I. Dimitrova, On nonlinear dynamics of interacting populations: Coupled kink waves in a system of two populations, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 2379–2388.
[43] N.K. Vitanov, I.P. Jordanov, Z.I. Dimitrova, On nonlinear population waves, Appl. Math. Comput. 215 (2009) 2950–2964.
[44] N.K. Vitanov, Z.I. Dimitrova, K.N. Vitanov, Traveling waves and statistical distributions connected to systems of interacting populations, Comput. Math. Appl. 66 (2013) 1666–1684.
[45] N.K. Vitanov, K.N. Vitanov, Population dynamics in presence of state dependent fluctuations, Comput. Math. Appl. 68 (2013) 962–971.
[46] A. Schubert, W. Glänzel, A dynamic look at a class of skew distributions. A model with scientometric application, Scientometrics 6 (1984) 149–167.
[47] N.K. Vitanov, K.N. Vitanov, Box model of migration channels, Math. Social Sci. 80 (2016) 108–114.
[48] J.O. Irwin, The place of mathematics in medical and biological sciences, J. R. Stat. Soc. 126 (1963) 1–44.
[49] J.O. Irwin, The generalized waring distribution applied to accident theory, J. R. Stat. Soc. 131 (1968) 205–225.
[50] V. Diodato, Dictionary of Bibliometrics, Haworth Press, Binghampton, NY, 1994.
[51] W.-C. Chen, On the weak form of the Zipf's law, J. Appl. Probab. 17 (1980) 611–622.
[52] H.A. Simon, On a class of skew distribution functions, Biometrica 42 (1955) 425–440.
[53] J.O. Irvin, The place of mathematics in medical and biological statistics, J. Roy. Statist. Soc. Ser. A 126 (1963) 1–45.
[54] G. Herdan, Quantitative Linguistics, Butterworths, London, 1964.
[55] R.E. Wyllys, Empirical and theoretical bases of Zipf's law, Libr. Trends 30 (1981) 53–64.
[56] J. Tague, The success-breeds-success phenomenon and bibliometric processes, J. Amer. Soc. Inf. Sci. 32 (1981) 280–286.
[57] N.K. Vitanov, Science Dynamics and Research Production. Indicators, Indexes, Statistical Laws, and Mathematical Models, Springer Internationsl Publishing, Switzerland, 2016.