Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 264 (2015) 138–159

FUZZY sets and systems

www.elsevier.com/locate/fss

OM3: Ordered maxitive, minitive, and modular aggregation operators – Axiomatic and probabilistic properties in an arity-monotonic setting

Anna Cena [a,b], Marek Gagolewski [a,c,*]

[a] *Systems Research Institute, Polish Academy of Sciences, ul. Newelska 6, 01-447 Warsaw, Poland*
[b] *Interdisciplinary PhD Studies Program, Institute of Computer Science, Polish Academy of Sciences, ul. J. Kazimierza 5, 02-248 Warsaw, Poland*
[c] *Faculty of Mathematics and Information Science, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warsaw, Poland*

Received 30 December 2013; received in revised form 9 March 2014; accepted 2 April 2014

Available online 12 April 2014

Abstract

The recently-introduced OM3 aggregation operators fulfill three appealing properties: they are simultaneously minitive, maxitive, and modular. Among the instances of OM3 operators we find e.g. OWMax and OWMin operators, the famous Hirsch h-index and all its natural generalizations. In this paper the basic axiomatic and probabilistic properties of extended, i.e. in an arity-dependent setting, OM3 aggregation operators are studied. We illustrate the difficulties one is inevitably faced with when trying to combine the quality and quantity of numeric items into a single number. The discussion on such aggregation methods is particularly important in the information resources producers assessment problem, which aims to reduce the negative effects of information overload. It turns out that the Hirsch-like indices of impact do not fulfill a set of very important properties, which puts the sensibility of their practical usage into question. Moreover, thanks to the probabilistic analysis of the operators in an i.i.d. model, we may better understand the relationship between the aggregated items' quality and their producers' productivity.

Keywords: Aggregation; Ordered modularity, maxitivity and minitivity; Arity-monotonicity; Impact assessment; Hirsch's h-index; Informetrics

1. Introduction

Informetrics is an active and important research field that deals with measurable aspects of computer and information science. Informetric methods aim to:

- accurately model and qualitatively explain various information processes-related phenomena (like information flow or growth in time),

* Corresponding author.
E-mail address: gagolews@ibspan.waw.pl (M. Gagolewski).

http://dx.doi.org/10.1016/j.fss.2014.04.001

- reliably evaluate the quality or measure the performance of information items and their producers, and
- efficiently manage digital libraries or information storage centers.

Companies, governments, web bots and individuals all over the world generate huge amounts of data of various kinds. It is known that, for example, the World Wide Web size grows exponentially. The number of on-line social networking services accounts often exceeds hundreds of millions (e.g. *Facebook*: 1.1 billion, *Twitter*: 0.5 billion). Each active user is a "producer" of new information items that are "assessed" by the members of the on-line community (cf. e.g. "Like", "Share", or "Follow" buttons).

A similar behavior may be observed in the system of Java, Python, or R software packages/libraries. For instance, the number of items on the CRAN (Comprehensive R Archive Network) repository reached over 5270 items in March 2014, and this number is expected to double within the next 3–4 years.

What is more, one may find the same pattern in digital libraries. The number of new scholarly articles also grows at a rapid pace. Elsevier's *Scopus* currently archives over 53.3 million items, of which ∼2.5 million are those added in the year 2013. Of course, the true number of papers ever published is much higher, as none of the databases has full content coverage.

It is evident that each user (either a human or a virtual one) of the above-mentioned data banks is likely to suffer from a so-called information overload. This implies an urgent need for development of:

- valid methods for automated quality management (e.g. that may indicate which items are worth being examined),
- multicriteria decision making techniques (e.g. in order to react effectively to some events), and most importantly,
- new ways to synthetically express various characteristics of information processes being studied (e.g. so that they may be understood and/or organized in an effective way).

Among such methods one may find e.g. the famous Hirsch's h-index [32] or other so-called informetric indices of impact, cf. [2], which usage and recognition is – quite unfortunately – often reduced to the domain of scientometrics, see [16] for one of a few notable exceptions to this rule. Even though such tools are defined in a very simple – if not trivial at a first glance – way, they are currently subjects of very intensive, yet still far from being advanced, theoretical research at the intersection of computer science, applied mathematics and operational research. It is because a satisfactory answer to questions like what in fact do they measure, where they can be applied, what are their formal properties, and so on, has not been provided yet.

Principally, these tools – despite the controversies about their usage in particular applications, like in the evaluation of science – are instances of very theoretically interesting mathematical objects. For example, the above-mentioned h-index is just one of uncountably many so-called aggregation operators, cf. [6,29], i.e. functions that map the space of vectors with elements in $\mathbb{I}$ into $\mathbb{I} := [a, b]$ or, in the language of probability, sample statistics, understood as functions of random variables.

The investigation of their foundations, properties and limitations is thus a very important task from the theoretical viewpoint. The classical approach in the theory of aggregation deals with the analysis and construction of functions that synthesize a numeric vector into a single number representing its "typical" value or some kind of "central tendency" of the given data object, see e.g. [29–31]. Most often, a domain of vectors of fixed size is assumed. In the discussed case, however, such a simplification is not valid, cf. also [26].

Let $P = \{p_1, \ldots, p_k\}$ be a set of k (abstract) producers. The i-th producer outputs n_i (abstract) products. Additionally, each product has been given some kind of quantitative valuation, rating, or assessment, e.g. concerning its overall quality. Consequently, in the most basic model, the state of p_i may be described by a sequence $\mathbf{x}^{(i)} = (x_1^{(i)}, \ldots, x_{n_i}^{(i)})$ with elements in $\mathbb{I}$, most often with $\mathbb{I} = [0, \infty]$, see Fig. 1 for an illustration. Table 1 lists some typical, yet highly interesting instances of such a situation, cf. e.g. [16,23,33]. The majority of them concerns the producers of information resources of various kinds.

Most importantly, one should note that the numbers of products may vary from producer to producer. The main aim of the *Producers Assessment Problem* (PAP, cf. [23]) is to construct methods for quantitative (numerical) assessment of producers. These mathematical or computational tools must necessarily meet only some mild assumptions. They shall somehow take into account the two following aspects of a producer's widely-conceived quality:

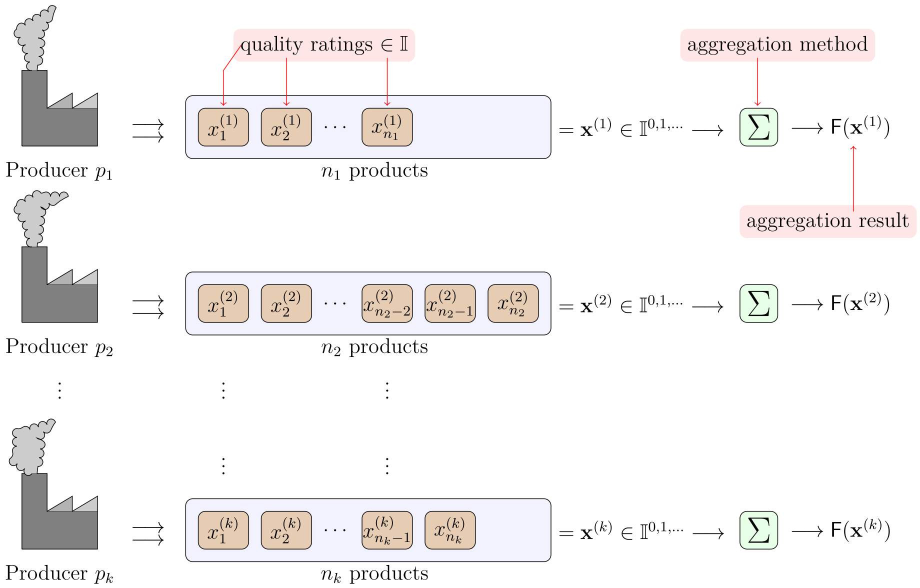

Fig. 1. Producers Assessment Problem.

Table 1
Typical instances of the Producer Assessment Problem (PAP).

Producer	Products	Rating method
R package author	R packages	Number of dependencies
Developer team	Python packages	Number of namespace imports from other projects
Web server	Web pages	Number of targeting web-links or Page Rank
Web service server	JSON/XML-RPC methods	Number of remote procedure calls
Developer team	Code repository (git, svn, etc.)	Number of commits
Publisher	On-line document	Number of downloads
Social networking profile	Posts	Number of "tweets" or "likes"
StackOverflow users	Answers to other users' questions	Up-votes
YouTube channels	Videos	Number of views
Digital library	Subscriber	Number of accesses
Scientist	Scientific articles	Number of citations
Scientific institute	Scientists	The *h*-index
Factory	Model-ranges of products	Sale results
Factory product	Supplied lots	Number of items without defects
Artist	Paintings	Auction price

1. its ability to output highly-valuated products, and
2. its overall productivity.

Moreover, it is often assumed that an increment in a product's quality, or an addition (concatenation) of a new product to the sequence, must not result in a decrease in the producer's rating. These properties are called nondecreasingness and arity-monotonicity, respectively.

In this paper we postulate a list of important axiomatic properties that aggregation operators for PAP should possess. The discussion is focused on a recently-introduced class of aggregation operators called OM3, see [9,18]. This family of functions is particularly interesting and worth being examined, as each OM3 operator fulfills three important and appealing properties: maxitivity, minitivity, and modularity, which generalizes additivity. Moreover, we will see that it naturally generalizes many Hirsch-based indices of impact and that they are connected with the universal [35] and – in some cases – the Sugeno integral [46].

Moreover, even though aggregation operators also appear in probability (under a name "statistics"), very rarely their stochastic properties, even in most basic i.i.d. models, are discussed. As such results are important both for theory as well as for practice, the OM3 operators will also be studied from this perspective. Note that some preliminary results have already been obtained (via Monte Carlo simulation studies) in [10] and for a particular subclass of OM3 in [22]. However, we shall observe that in an arity-monotonic setting these functions may behave in a nonstandard way. The new results presented in this paper may help to better understand the relationship between a producer's ability to output valuable goods and its productivity.

The paper is organized as follows. In the next section, the definition of the OM3 operators is presented and the most notable instances of this class of aggregation operators are introduced. In Section 3 we recall existing and also postulate new axiomatic properties useful in the Producers Assessment Problem, and characterize OM3 operators fulfilling them. Basic probabilistic properties, also in an arity-dependent setting, of OM3 operators are discussed in Section 4. Finally, Section 5 concludes the paper and indicates some open problems for future work.

2. The OM3 aggregation operators

From now on let $\mathbb{I} = [0, b]$ for some $b > 0$, possibly with $b = \infty$. Moreover, let $\mathbb{I}^{0,1,\cdots} = \bigcup_{n=0}^{\infty} \mathbb{I}^n$, with convention $\mathbb{I}^0 = \{\varnothing\}$, where $\varnothing$ denotes a 0-tuple (i.e. an empty vector, ()). In other words, $\mathbb{I}^{0,1,\cdots}$ is the set of vectors of arbitrary length with elements in $\mathbb{I}$. Additionally, let $[n] = \{1, 2, \ldots, n\}$ and $(n * x) = (x, x, \ldots, x) \in \mathbb{I}^n$.

Let us introduce the following class of (extended, i.e. defined for any sample size n) aggregation operators. For each such a function $\mathsf{F} : \mathbb{I}^{0,1,\cdots} \to \mathbb{I}$ we will assume the lower bound $\mathsf{F}(\varnothing) = 0$.

Definition 1. (See [9,18].) The OM3 operator $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$ generated by a sequence of mappings $\mathbf{w} = (\mathsf{w}_1, \mathsf{w}_2, \ldots)$, $\mathsf{w}_i : \mathbb{I} \to \mathbb{I}$, and a triangle of coefficients $\triangle = (c_{i,n})_{i\in[n],n\in\mathbb{N}}$, $c_{i,n} \in \mathbb{I}$, is given by

$$\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \bigvee_{i=1}^{n} \mathsf{w}_n(x_{(n-i+1)}) \wedge c_{i,n},$$

where $\mathbf{x} \in \mathbb{I}^n$ and $x_{(i)}$ denotes the i-th smallest value in $\mathbf{x}$.

It is easily seen that each OM3 operator is symmetric, i.e. its value is independent of the ordering of elements in $\mathbf{x}$, cf. [29, Proposition 2.34]. As most often in the aggregation theory we assume that a given aggregation operator is nondecreasing in each variable, the following result is worth recalling, see [9, Lemma 1] for the proof. Here, if necessary, we apply the convention that $\mathsf{w}_n(\infty) = \lim_{x\to\infty} \mathsf{w}_n(x)$.

Lemma 2 *(Reduction). (See [9].)* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is nondecreasing in each variable if and only if there exist* $\mathbf{w}' = (\mathsf{w}'_1, \mathsf{w}'_2, \ldots)$, $\mathsf{w}'_i : \mathbb{I} \to \mathbb{I}$, *and a triangle of coefficients* $\triangledown = (c'_{i,n})_{i\in[n],n\in\mathbb{N}}$ *satisfying the following conditions:*

(i) $(\forall n)$ w'_n *is nondecreasing,*
(ii) $(\forall n)$ $c'_{1,n} \le c'_{2,n} \le \cdots \le c'_{n,n}$,
(iii) $(\forall n)$ $0 \le \mathsf{w}'_n(0) \le c'_{1,n}$,
(iv) $(\forall n)$ $\mathsf{w}'_n(b) = c'_{n,n} \le b$,

such that $(\forall \mathbf{x})$ $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{M}_{\triangledown,\mathbf{w}'}(\mathbf{x})$.

Please be advised that the above "reduction" lemma implies that one may consider nondecreasing OM3 operators – with no loss in generality – only in the form provided above. In such a case, $\mathsf{M}_{\triangle,\mathbf{w}}$ is a classical aggregation function for $n > 1$ if and only if $\mathsf{w}_n(0) = 0$ and $\mathsf{w}_n(b) = b$, and for $n = 1$ iff $\mathsf{w}_n(x) = x$, see [29]. Also, if $\mathsf{w}_n(x) = x$, then an OM3 operator is idempotent, see also Section 3. However, we do not necessarily assume these conditions in this paper, as in the PAP case they are too restrictive: for fixed n and any $\mathbf{x} \in \mathbb{I}^n$ we only necessarily have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \in [\mathsf{w}_n(0), \mathsf{w}_n(b)]$.

The name OM3 comes from the three important properties that these aggregation operators fulfill. In [18] it has been shown that each such an operator (and no other) is ordered (symmetric) minitive, i.e. $(\forall n)$ $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)$ $\mathsf{F}(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) = \mathsf{F}(\mathbf{x}) \wedge \mathsf{F}(\mathbf{y})$, maxitive, i.e. $(\forall n)$ $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)$ $\mathsf{F}(\mathbf{x} \overset{S}{\vee} \mathbf{y}) = \mathsf{F}(\mathbf{x}) \vee \mathsf{F}(\mathbf{y})$, and modular, i.e. $(\forall n)$ $(\forall \mathbf{x}, \mathbf{y} \in \mathbb{I}^n)$

$\mathsf{F}(\mathbf{x} \overset{S}{\vee} \mathbf{y}) + \mathsf{F}(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) = \mathsf{F}(\mathbf{x}) + \mathsf{F}(\mathbf{y})$, where $\mathbf{x} \overset{S}{\vee} \mathbf{y} = (x_{(1)} \vee y_{(1)}, \ldots, x_{(n)} \vee y_{(n)})$ and $\mathbf{x} \overset{S}{\wedge} \mathbf{y} = (x_{(1)} \wedge y_{(1)}, \ldots, x_{(n)} \wedge y_{(n)})$. This is because for all $\mathbf{x} \in \mathbb{I}^{0,1,\cdots}$ it holds:

$$\begin{aligned}\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) &= \bigvee_{i=1}^{n} \mathsf{w}_n(x_{(n-i+1)}) \wedge c_{i,n} \\ &= \bigwedge_{i=1}^{n} \big(\mathsf{w}_n(x_{(n-i+1)}) \vee c_{i-1,n}\big) \wedge c_{n,n} \\ &= \sum_{i=1}^{n} \big(\big(\mathsf{w}_n(x_{(n-i+1)}) \vee c_{i-1,n}\big) \wedge c_{i,n} - c_{i-1,n}\big),\end{aligned}$$

with convention $c_{0,n} = 0$, see [18, Theorem 20] for the proof.

Remark 3. Some OM3 operators correspond to the very general family of universal integrals, which were defined in [35] (see also [24,47] and [4,8] for other uses of fuzzy measures and integrals in bibliometrics).

Fix $n > 0$. Consider a measurable space $([n], 2^{[n]})$, where $2^{[n]}$ denotes the power set of $\{1, \ldots, n\}$. Additionally, let $\mu : 2^{[n]} \to \mathbb{I}$ be a discrete monotone measure, i.e. a function such that $\mu(\emptyset) = 0$, $\mu(\mathbb{I}^n) > 0$, and for which for all $U, V \in 2^{[n]}$ it holds $U \subseteq V \Longrightarrow \mu(U) \leq \mu(V)$. Note that μ is not necessarily additive.

Additionally, for any $\mathbf{x} \in \mathbb{I}^n$ let $\mathsf{h}^{(\mu,\mathbf{x})}(t) = \mu(\{i : x_{(n-i+1)} \geq t\})$, $t \in \mathbb{I}$, i.e. the measure of the t-level set of $\mathbf{x}$. Do note that as in fact we only apply μ on subsets like $U = \{1, 2, \ldots, k\}$, with no loss in generality we may assume that μ is generated by a nondecreasing function of the counting measure, $\mu(U) = \varphi(|U|) = \varphi(k)$, with $k = 0$ case corresponding to $\emptyset$, i.e. $\varphi : \{0, 1, \ldots, n\} \to \mathbb{I}$. Thus, μ is a symmetric measure.

First of all, it may be observed that the Sugeno integral [46] given by $\mathcal{S}(\mu, \mathbf{x}) = \sup_{t\in\mathbb{I}}\{t \wedge \mathsf{h}^{(\mu,\mathbf{x})}(t)\}$ is equivalent to $\mathsf{M}_{\mathbf{w},\triangle}(\mathbf{x}) = \bigvee_{i=1}^{n} x_{(n-i+1)} \wedge \varphi(i)$. Thus, changing the coefficients from $\triangle$ corresponds to transforming the monotone measure μ.

More generally, it may be shown that we have $\mathcal{I}(\mu, \mathbf{x}) = \sup_{t\in\mathbb{I}}\{\mathsf{w}_n(t) \wedge \mathsf{h}^{(\mu,\mathbf{x})}(t)\} = \bigvee_{i=1}^{n} \mathsf{w}_n(x_{(n-i+1)}) \wedge \varphi(i)$. If $\mathsf{w}_n(0) = 0$ and $\mathsf{w}_n(b) = b$, then such a function is a universal integral generated by a pseudo-multiplication operator $\otimes$ (see [35]) such that $u \otimes v = \mathsf{w}_n(u) \wedge v$, and of course generalizes the Sugeno integral. □

It turns out that in some cases an OM3 operator may be written in different, often simpler, forms.

Lemma 4. *Let $\mathsf{M}_{\triangle,\mathbf{w}}$ be a nondecreasing OM3 operator with continuous w_n for all n. Then $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{w}_n(\bigvee_{i=1}^{n} x_{(n-i+1)} \wedge c'_{i,n})$, where $c'_{i,n} = \mathsf{w}_n^{-1}(c_{i,n}) = \sup\{x : \mathsf{w}_n(x) \leqslant c_{i,n}\}$.*

Proof. Without loss of generality let $\mathbf{w}$ and $\triangle$ be of the form given in Lemma 2. Fix $n \geq 1$ and $\mathbf{x} \in \mathbb{I}^n$. Moreover, let i be such that $\mathsf{M}_{\mathbf{w},\triangle}(\mathbf{x}) = \mathsf{w}_n(x_{(n-i+1)}) \wedge c_{i,n}$.

Let $y \in \operatorname{range} \mathsf{w}_n = [\mathsf{w}_n(0), \mathsf{w}_n(b)]$. Take any $y \in \operatorname{range} \mathsf{w}_n$ and let $S_y = \{x : \mathsf{w}_n(x) = y\}$. This implies $\mathsf{w}_n(\mathsf{w}_n^{-1}(y)) = \mathsf{w}_n(\sup S_y) = \mathsf{w}_n(\max S_y) = y$, as S_y is a closed set under continuity of w_n.

Additionally, note that under our assumptions $c_{j,n} \in \operatorname{range} \mathsf{w}_n$ for all j. Thus, by monotonicity of w_n implying the fact that for all d, e it holds $\mathsf{w}_n(d \wedge e) = \mathsf{w}_n(d) \wedge \mathsf{w}_n(e)$, we have $\mathsf{w}_n(x_{(n-i+1)}) \wedge c_{i,n} = \mathsf{w}_n(x_{(n-i+1)}) \wedge \mathsf{w}_n(\mathsf{w}_n^{-1}(c_{i,n})) = \mathsf{w}_n(x_{(n-i+1)} \wedge \mathsf{w}_n^{-1}(c_{i,n}))$, and the proof is complete. □

Remark 5. On the other hand, if w_n is not continuous, then a counterexample for the antecedent of the above theorem may easily be constructed. For instance, let $\mathsf{w}_1(x) = \lfloor x \rfloor$, $c_{1,1} = 0.5$, and $\mathbf{x} = (1.5)$. Then $\mathsf{w}_1(1.5) \wedge 0.5 = 1 \wedge 0.5 = 0.5$ and $\mathsf{w}_1(\mathsf{w}_1^{-1}(\mathsf{w}_1(1.5)) \wedge \mathsf{w}_1^{-1}(0.5)) = \mathsf{w}_1(1 \wedge 1) = 1$. □

Moreover, further on we will need the following auxiliary result. It is a slightly generalized version of [10, Proposition 1].

Lemma 6. *Let $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \bigvee_{i=1}^{n} \mathsf{w}_n(x_{(n-i+1)}) \wedge c_{i,n}$, where $(\forall n)$ $\mathsf{w}_n : \mathbb{I} \to \mathbb{I}$ is nondecreasing and $c_{1,n} < \cdots < c_{n,n}$. Then $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$ is a nondecreasing OM3 operator if and only if for any n there exists a strictly increas-*

ing function $\mathsf{f}_n : \mathbb{I} \to \mathbb{I}$ *and a nondecreasing function* $\mathsf{w}'_n : \mathbb{I} \to \mathbb{I}$, *such that for all* $\mathbf{x} \in \mathbb{I}^n$ *it holds* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{f}_n(\bigvee_{i=1}^n (\mathsf{w}'_n(x_{(n-i+1)}) \wedge i))$.

Proof. ($\Longrightarrow$) Fix n. Let f_n be a piecewise linear continuous function such that we have $\mathsf{f}_n(i) = c_{i,n}$ for all $i \in [n]$. It is obvious that f_n is a strictly increasing function, since the sequence $(c_{i,n})_{i\in[n]}$ is strictly increasing, and f_n is onto $\mathbb{I}$. Hence, there exists its (also strictly increasing) inverse, f_n^{-1}, for which we have $\mathsf{f}_n^{-1}(c_{i,n}) = i$. Thus, it holds $\mathsf{f}_n^{-1}(\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})) = \bigvee_{i=1}^n (\mathsf{f}_n^{-1}(\mathsf{w}_n(x_{(n-i+1)})) \wedge \mathsf{f}_n^{-1}(c_i)) = \bigvee_{i=1}^n ((\mathsf{f}_n^{-1} \circ \mathsf{w}_n)(x_{(n-i+1)}) \wedge i)$ for any $\mathbf{x} \in \mathbb{I}^n$. We may therefore set $\mathsf{w}'_n = \mathsf{f}_n^{-1} \circ \mathsf{w}_n$ and we have $\mathsf{f}_n(\bigvee_{i=1}^n (\mathsf{w}'_n(x_{(n-i+1)}) \wedge i)) = \mathsf{M}_{\mathsf{w},\triangle}(\mathbf{x})$.

($\Longleftarrow$) We have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{f}_n(\bigvee_{i=1}^n (\mathsf{w}'_n(x_{(n-i+1)}) \wedge i)) = \bigvee_{i=1}^n (\mathsf{f}_n \circ \mathsf{w}'_n)(x_{(n-i+1)}) \wedge \mathsf{f}_n(i)$. It is easily seen that $\mathsf{w}_n := \mathsf{f}_n \circ \mathsf{w}'_n$ is nondecreasing (since f_n is strictly increasing and w'_n is nondecreasing) and $c_{i,n} := \mathsf{f}_n(i) < \mathsf{f}_n(i+1) = c_{i+1,n}$. Please note that for $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$ such that $\mathbf{x} \leqslant \mathbf{y}$ we have $(\forall i)$ $\mathsf{w}_n(x_{(n-i+1)}) \wedge c_{i,n} \leqslant \mathsf{w}_n(y_{(n-i+1)}) \wedge c_{i,n}$. Thus, $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \leqslant \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y})$, which completes the proof. □

Remark 7. Please note that the assumption $c_{1,n} < c_{2,n} < \cdots < c_{n,n}$ in Lemma 6 cannot be weakened. For example, let us consider $n = 3$ and an OM3 operator given by $\mathsf{M}_{\triangle,\mathsf{w}}(\mathbf{x}) = \bigvee_{i=1}^n x_{(n-i+1)} \wedge c_{i,n}$, with $c_{1,3} = c_{2,3} = 1$ and $c_{3,3} = 2$.

We have $\mathsf{M}_{\triangle,\mathsf{w}}(0, 0, 0) = 0$. Thus, $\mathsf{f}_n(\mathsf{w}'_n(0)) \wedge \mathsf{f}_n(3) = 0$, and as $\mathsf{f}_n(3) > 0$, it follows that $\mathsf{f}_n(\mathsf{w}'_n(0)) = 0$.

Moreover, $\mathsf{M}_{\triangle,\mathsf{w}}(1.5, 0, 0) = \mathsf{M}_{\triangle,\mathsf{w}}(1.5, 1.5, 0) = 1$. Consequently, $(\mathsf{f}_n(\mathsf{w}'_n(1.5)) \wedge \mathsf{f}_n(1)) \vee (\mathsf{f}_n(\mathsf{w}'_n(0)) \wedge \mathsf{f}_n(3)) = \mathsf{f}_n(\mathsf{w}'_n(1.5)) \wedge \mathsf{f}_n(1) = 1$ and $(\mathsf{f}_n(\mathsf{w}'_n(1.5)) \wedge \mathsf{f}_n(1)) \vee (\mathsf{f}_n(\mathsf{w}'_n(0)) \wedge \mathsf{f}_n(3)) = \mathsf{f}_n(\mathsf{w}'_n(1.5)) \wedge \mathsf{f}_n(2) = 1$. This gives $\mathsf{f}_n(\mathsf{w}'_n(1.5)) = 1$ and $\mathsf{f}_n(1) \geq 1$, since $\mathsf{f}_n(1) < \mathsf{f}_n(2)$. However, $\mathsf{M}_{\triangle,\mathsf{w}}(1.5, 1.5, 1.5) = 1.5$ implies $\mathsf{f}_n(\mathsf{w}'_n(1.5)) \wedge \mathsf{f}_n(3) = 1 \wedge \mathsf{f}_n(3)$. Therefore, we get $1 \wedge \mathsf{f}_n(3) \neq 1.5$ and, as $\mathsf{f}_n(3) > 1$, we obtain a contradiction. □

We easily see that special cases of the OM3 operators include OWMax/OWMin [13,14], all order statistics, as well as all transformations of OWMax operators concordant with Theorem 4.

Also, considering the PAP context where $\mathbf{x}$ represents quality measures of products output by some producer and $\mathbb{I} = [0, \infty]$, we may obtain e.g. the total number of products (if $\mathsf{w}_n(x) = n$ for all x and $c_{1,n} = \cdots = c_{n,n} = n$) and the total number of products of non-zero quality (if $\mathsf{w}_n(0) = 0$ and $\mathsf{w}_n(x) = \infty$ for all $x > 0$ and $c_{i,n} = i$). Moreover, the famous h-index by J.E. Hirsch [32], the generalized h-index given by $\widetilde{\mathsf{H}}(\mathbf{x}) = \bigvee_{i=1}^n x_{(n-i+1)} \wedge i$ (an OM3 operator of perhaps the simplest possible form), as well as their many modifications, like the $h^{(2)}$-index [36] and other ones similar to those defined in [12], also fall into this class:

Lemma 8. *Let* $\mathsf{w} : \mathbb{I} \to \mathbb{I}$ *be a nondecreasing function. Then*

$$\mathsf{H}_\mathsf{w}(\mathbf{x}) = \max\{i = 0, 1, \ldots, n : \mathsf{w}(x_{(n-i+1)}) \geq i\},$$

with convention $x_{(n+1)} = 0$, *is a nondecreasing OM3 operator.*

Proof. Fix n, $\mathbf{x} \in \mathbb{I}^n$ and let $j = \mathsf{H}_\mathsf{w}(\mathbf{x}) \in \{0, 1, \ldots, n\}$. If $j > 0$, then it is easily seen that $j = \lfloor \mathsf{w}(x_{(n-j+1)}) \rfloor \wedge j$, as $\mathsf{w}(x_{(n-j+1)}) \geq j$. Moreover, for all $i < j$ by the nondecreasingness of w we have $\lfloor \mathsf{w}(x_{(n-i+1)}) \rfloor \wedge i < j$ and for $i > j$ it holds $\lfloor \mathsf{w}(x_{(n-i+1)}) \rfloor \wedge i \leq j$ as otherwise j would not be as defined. On the other hand, if $j = 0$ then surely $\mathsf{w}(x_{(n)}) < 1$ and we have $\lfloor \mathsf{w}(x_{(i)}) \rfloor$ for all i.

Thus, $\mathsf{H}_\mathsf{w}(\mathbf{x}) = \bigvee_{i=1}^n \lfloor \mathsf{w}(x_{(n-i+1)}) \rfloor \wedge i$, i.e. it is a nondecreasing OM3 operator with $\mathsf{w}_n = \mathsf{w}$ and $c_{i,n} = i$ for all i, n. □

What is even more, by applying some input vector transformations we may obtain other indices of impact, like the g-index [15], defined as $\mathsf{G}(\mathbf{x}) = \max\{i = 0, 1, \ldots, n : \sum_{j=1}^i x_{(n-j+1)} \leq i^2\}$, and w-index [50], given by $\mathsf{W}(\mathbf{x}) = \max\{i = 0, 1, \ldots, n : x_{(n-i+1)} \leq n - i + 1\}$. It is because for any nonincreasingly sorted $\mathbf{x}$ we have:

$$\mathsf{G}(\mathbf{x}) = \mathsf{M}_{\mathbf{w},\triangle}\big(0 \vee \mathtt{cummin}\big(\mathtt{cumsum}(\mathbf{x}) - \big(1^2, 2^2, \ldots, n^2\big) + (1, 2, \ldots, n)\big)\big),$$

$$\mathsf{W}(\mathbf{x}) = \mathsf{M}_{\mathbf{w},\triangle}\big(\mathtt{cummin}\big(\mathbf{x} + (1, 2, \ldots, n) - 1\big)\big),$$

where $\mathsf{w}_n(x) = \lfloor x \rfloor$, $c_{i,n} = i$ for all i, n, $\mathtt{cummin}, \mathtt{cumsum} : \mathbb{I}^{0,1,\cdots} \to \mathbb{I}^{0,1,\cdots}$ denote the symmetrized cumulative minimum and sum, respectively, i.e.:

$$\mathtt{cummin}(\mathbf{x}) = (x_{(n)}, x_{(n)} \wedge x_{(n-1)}, x_{(n)} \wedge x_{(n-1)} \wedge x_{(n-2)}, \ldots),$$
$$\mathtt{cumsum}(\mathbf{x}) = (x_{(n)}, x_{(n)} + x_{(n-1)}, x_{(n)} + x_{(n-1)} + x_{(n-2)}, \ldots),$$

and operations $+$, $-$, and $\vee$ applied on vectors are performed element-wise. Both facts have been proven in [24].

3. Axiomatic properties of OM3 operators

In this section we discuss the most important properties that are of particular interest in the Producers Assessment Problem framework. Let us begin the review of notable axiomatic properties with conditions that do not depend on input vector's arities. In other words, the following properties assume that only vectors of the same lengths are considered at a time, see [21] for discussion.

3.1. Notable arity-free properties

We have already considered a few arity-free properties: symmetry, symmetric modularity, minitivity, maxitivity, which are fulfilled by all OM3 operators, and finally nondecreasingness in Lemma 2. Nondecreasingness of an aggregation operator is a sine qua non condition for the majority of practical applications. Sometimes, however, one may require a stronger condition guaranteeing the sensitivity of a function for a change of an input vector's elements. First of all, one may wish that an aggregation operator's value always increases its value on any possible input vector's element incrementation. However, it turns out that for some vectors it is always impossible.

Proposition 9. *Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 2. Then* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is surely not* ***strictly increasing*** *(sensitive, cf. [29, Proposition 2.4]), i.e. there always exist* n, $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$, $\mathbf{x} \le \mathbf{y}$ *and* $\mathbf{x} \ne \mathbf{y}$ *(denoted* $\mathbf{x} <_1 \mathbf{y}$*), such that* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \not< \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y})$.

Proof. Fix n and take any $x, y \in \mathbb{I}$, $x < y \le b$. Recall that $(n * x) = (x, x, \ldots, x) \in \mathbb{I}^n$. We have $\mathsf{M}_{\triangle,\mathbf{w}}(x, (n-1)*0) = \mathsf{w}_n(x) \wedge c_{1,n}$ and $\mathsf{M}_{\triangle,\mathbf{w}}(y, (n-1)*0) = \mathsf{w}_n(y) \wedge c_{1,n}$. Thus, $\mathsf{M}_{\triangle,\mathbf{w}}(x, (n-1)*0) < \mathsf{M}_{\triangle,\mathbf{w}}(y, (n-1)*0)$ if and only if $c_{1,n} = \mathsf{w}_n(b)$ and $\mathsf{w}_n(x) < \mathsf{w}_n(y)$. However, now we have $\mathsf{w}_n(x) = \mathsf{M}_{\triangle,\mathbf{w}}(x, (n-1)*0) = \mathsf{M}_{\triangle,\mathbf{w}}(n*x) = \mathsf{w}_n(x)$, and the proof is complete. □

Thus, we may be interested in a weaker form of sensitivity, which may be formulated as follows.

Proposition 10. *Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 2. Then* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is* ***weak-sensitive****, i.e. for all* n, $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$ *if* $\mathbf{x} < \mathbf{y}$*, then* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) < \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y})$*, if and only if* $(\forall n)$ w_n *is strictly increasing and* $(\forall x \in (0, b))$ *there exists* $i \in [n]$ *such that* $\mathsf{w}_n(x) \in [c_{i-1,n}, c_{i,n})$*, with convention* $c_{0,n} = 0$.

Proof. ($\Longrightarrow$) Fix n. First note that if there exist $x < x'$ such that $\mathsf{w}_n(x) = \mathsf{w}_n(x')$ then $\mathsf{w}_n(x) = \mathsf{w}_n(x) \wedge c_{n,n} = \mathsf{M}_{\triangle,\mathbf{w}}(n*x) = \mathsf{M}_{\triangle,\mathbf{w}}(n*x') = \mathsf{w}_n(x') \wedge c_{n,n} = \mathsf{w}_n(x')$, a contradiction. Thus, w_n must necessarily be strictly increasing.

Take any $0 < x < y < b$ and assume that there exists $i < n$ such that $\mathsf{w}_n(x) < c_{i,n}$ and $\mathsf{w}_n(y) \ge c_{i,n}$. Then for all $\varepsilon > 0$ we have $\mathsf{M}_{\triangle,\mathbf{w}}(i*y, (n-i)*0) = c_{i,n} = \mathsf{M}_{\triangle,\mathbf{w}}(i*(y+\varepsilon), (n-i)*x)$, which again leads to a contradiction. It implies that for every i it either holds $(\forall x \in (0, b))$ $\mathsf{w}_n(x) < c_{i,n}$ or $(\forall x \in (0, b))$ $\mathsf{w}_n(x) \ge c_{i,n}$.

($\Longleftarrow$) Fix n and take any $\mathbf{x}, \mathbf{y} \in \mathbb{I}^n$ such that $\mathbf{x} < \mathbf{y}$. If $\mathbf{x} = (n*0)$, then $\mathsf{w}_n(y_{(1)}) > \mathsf{w}_n(0) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$, $c_{n,n} = \mathsf{w}_n(b) > \mathsf{w}_n(0)$, and the property of our interest indeed holds.

Let $\mathbf{x} \ne (n*0)$. We have two cases:

- if there exists j such that $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = c_{j,n}$, then $\mathsf{w}_n(x_{(n-j+1)}) \ge c_{j,n}$ and $x_{(1)} = 0$. But here $y_{(1)} > 0$, $c_{n,n} > w_n(y_{(1)}) > c_{j,n}$ and thus $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) < \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y})$;
- if there exists j such that $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{w}_n(x_{(n-j+1)})$, then $\mathsf{w}_n(x_{(n-j+1)}) < c_{j,n}$ and thus $\mathsf{w}_n(x_{(n-j+1)}) < \mathsf{w}_n(y_{(n-j+1)})$, which implies $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) < \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y})$.

Therefore, the proof is complete. □

Note that if w_n is continuous, then the weak sensitivity property holds if either $c_{i,n} = \mathsf{w}_n(0)$ or $c_{i,n} = \mathsf{w}_n(b)$. It may be seen that e.g. all sample quantiles do obey this property.

Continuity in some cases may be not only important, but may also lead to significant simplifications in the form of $\triangle$ and $\mathbf{w}$.

Proposition 11. *Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 2. Then* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is **continuous**, if and only if* $(\forall n)$ w_n *is continuous.*

Proof. $(\Longleftarrow)$ Trivial.

$(\Longrightarrow)$ Assume that for some n the function w_n is discontinuous at x^*, i.e. there exist a sequence $(x_m)_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty} x_m = x^*$ and $\lim_{x_m\to x^*} \mathsf{w}_n(x_m) \neq \mathsf{w}_n(x^*)$. Additionally, let $(\mathbf{x}^{(m)})_{m\in\mathbb{N}}$ be such that $\mathbf{x}^{(m)} = (n * x_m)$. We have $\lim_{m\to\infty} \mathbf{x}^{(m)} = \mathbf{x}^* = (n * x^*)$. However, $\lim_{\mathbf{x}^{(m)}\to\mathbf{x}^*} \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}^{(m)}) = \lim_{\mathbf{x}^{(m)}\to\mathbf{x}^*} \mathsf{M}_{\triangle,\mathbf{w}}(n * x_m) = \lim_{\mathbf{x}^{(m)}\to\mathbf{x}^*} \mathsf{w}_n(x_m) \wedge c_{n,n} = \lim_{\mathbf{x}^{(m)}\to\mathbf{x}^*} \mathsf{w}_n(x_m) \neq \mathsf{w}_n(x^*) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}^*)$. □

Note that according to [29, Proposition 2.8], continuity of a nondecreasing function $\mathsf{M}_{\triangle,\mathbf{w}}$ is equivalent to its continuity in each variable.

3.2. Notable arity-dependent properties

Up to now we have not discussed yet any properties that take into account the relationships between functions' values for vectors of different arities. As the very nature of the Producers Assessment Problem is such that each agent may output an arbitrary number of products, the following conditions are worth-noting.

Firstly, we should stress that arity-monotonicity is most often treated as the most fundamental condition in the PAP framework, see [12,17,21,23,40–44,49–51] and cf. [34,39]. This axiom implies that creating a new product never results in a decrease in a producer's valuation. Such a condition seems to be reasonable if one assumes that each product fulfills some "minimal quality requirements" (denoted with 0) and a producer should not be punished for producing such a "good", but perhaps not an outstanding, item. It may be observed that almost all the "modern", post-Hirsch indices of scientific impact (also known as performance indices) do obey this property.

Proposition 12. *(See [18].) Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 2. Then* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is **arity-monotonic**, i.e. such that for each* $\mathbf{x} \in \mathbb{I}^{0,1,\cdots}$ *it holds* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, 0) \geq \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$*, if and only if for all* $x \in \mathbb{I}$ *and* i, n *we have* $\mathsf{w}_1(x) \leq \mathsf{w}_2(x) \leq \cdots$ *and* $c_{i,n} \leq c_{i,n+1}$.

The proof is omitted. Nondecreasing, symmetric, and arity-monotonic aggregation operators are called e.g. impact functions in [23], pre-impact indices in [44], scientific impact indices in [50], or size-dependent indicators of scientific impact in [48]. The latter three are of course used by scientometricians.

Note that arity-monotonicity may not be proper in some practical situations. For example, it is impossible to "promote" producers of low productivity (e.g. young scientists in the bibliometric context) with an arity-monotonic OM3 operator. E.g. if $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \bigvee_{i=1}^{n} \frac{x_{(n-i+1)}}{\sqrt{n}} \wedge i$, then a producer with bigger productivity has to obtain better product quality than the one who is of small productivity. However, it is easily seen that such a function cannot fulfill the property discussed here.

Zero-insensitivity, see [21,50], pays special attention to the extension of an input vector by an element equal to 0, which is the minimal possible valuation that a product may gain. Some authors even assume that an aggregation operator must necessarily obey this property in PAP-like contexts, see [12,49,51]. Importantly, this condition is a quite natural way to embed the $\mathbb{I}^{0,1,\cdots}$ space into $\mathbb{I}^{\infty}$, see [24] for discussion and alternative suggestions.

Proposition 13. *(See [9].) Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 2. Then* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is **zero-insensitive**, i.e. such that for each* $\mathbf{x} \in \mathbb{I}^{0,1,\cdots}$ *it holds* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, 0) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$*, if and only if* $(\forall n)$ $(\forall i \in [n])$ $c_{i,n} = c_{i,n+1}$*, and*

(i) $w_1(0) = 0$,
(ii) *if* x *is such that* $\mathsf{w}_n(x) < c_{n,n}$*, then* $\mathsf{w}_n(x) = \mathsf{w}_{n+1}(x)$*,*
(iii) *if* x *is such that* $\mathsf{w}_n(x) = c_{n,n}$*, then* $\mathsf{w}_{n+1}(x) \geqslant c_{n,n}$.

See [9, Theorem 2] for the proof. Please note that the first condition is due to the fact that $\mathsf{M}_{\triangle,\mathbf{w}}(\varnothing) = 0$. The above proposition implies that OM3 operators may in such a case be generated by using a single nondecreasing function w and a *sequence* of coefficients $(c_1, c_2, \ldots)$. Moreover, it is easily seen that each nondecreasing and zero-insensitive aggregation operator is also arity-monotonic.

Please note that the only weak-sensitive and zero-insensitive OM3 operator is a strictly increasing function of the Max operator.

Later on it will turn out that zero-insensitivity and even arity-monotonicity may be problematic: as far as OM3 aggregation operators are concerned, these properties may sometimes lead to very simple functions only. Thus, they seem to be too restrictive. However, non-zero-insensitive OM3 operators may appear as artificial for practitioners.

F-insensitivity, see [21,50], cf. also "conservative productivity increment" in [39] and the notion of "stability" in [5], requires that the output value of an aggregation operator does not change when we add to the input vector $\mathbf{x}$ a value not greater than the current valuation of $\mathbf{x}$.

Proposition 14. *(See [9].) Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 2. Then* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is arity-monotonic and* ***F-insensitive****, i.e. such that for all* $\mathbf{x} \in \mathbb{I}^{0,1,\cdots}$ *and* $y \le \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$ *we have* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, y) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$*, if and only if there exists:*

(i) $w_1(0) = 0$,
(ii) *a nondecreasing function* w*, for which if there exists* x *such that* $\mathsf{w}(x) > x$*, then* $(\forall y \in [x, \mathsf{w}(x)])\ \mathsf{w}(y) = \mathsf{w}(x)$*,*
(iii) *a nondecreasing sequence* $(c_1, c_2, \ldots)$*, such that* $(\forall i)\ c_i \notin \{x \in \mathbb{I} : x < \mathsf{w}(x)\}$*,*

such that $\mathsf{w}_n = \mathsf{w} \wedge c_n$ *and* $c_{i,n} = c_i$*.*

See [9, Theorem 3] for the proof. Note that each F-insensitive, nondecreasing and arity-monotonic symmetric aggregation operator is also zero-insensitive. Among such aggregation operators we have e.g. the Hirsch index and Max.

In certain environments we may also want to require that the overall valuation may only change if we extend a vector by a value strictly greater than the maximal one. Let us assume that $\mathsf{Max}(\varnothing) = 0$. We have the following.

Proposition 15. *Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 2. Then* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is arity-monotonic and* ***max-insensitive****, i.e. such that for each* $\mathbf{x} \in \mathbb{I}^{0,1,\cdots}$ *it holds* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, \mathsf{Max}(\mathbf{x})) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$*, if and only if*

(i) $w_1(0) = 0$,
(ii) $(\forall n)\ (\forall x \in \mathbb{I})\ \mathsf{w}_n(x) = \mathsf{w}_1(x)$,
(iii) $(\forall n)\ (\forall i \in [n])\ c_{i,n} = \mathsf{w}_1(b)$.

Proof. ($\Longleftarrow$) We have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{w}_1(x_{(n)})$ with $\mathsf{M}_{\triangle,\mathbf{w}}(0) = 0$, which obviously is a max-insensitive aggregation operator.

($\Longrightarrow$) First note that as we have assumed $\mathsf{Max}(\varnothing) = 0$, then $\mathsf{M}_{\triangle,\mathbf{w}}(\varnothing, 0) = \mathsf{M}_{\triangle,\mathbf{w}}(0) = 0$ if $\mathsf{w}_1(0) = 0$.

Take any $x \in \mathbb{I}$ and $n \ge 1$. We have $\mathsf{w}_1(x) = \mathsf{w}_1(x) \wedge c_{1,1} = \mathsf{M}_{\triangle,\mathbf{w}}(x) = \mathsf{M}_{\triangle,\mathbf{w}}(x, x) = \cdots = \mathsf{M}_{\triangle,\mathbf{w}}(n * x) = \mathsf{w}_n(x) \wedge c_{n,n} = \mathsf{w}_n(x)$. Thus, $\mathsf{w}_n(x) = \mathsf{w}_1(x)$ and $c_{n,n} = \mathsf{w}_1(b)$.

Now let us assume that $x > 0$ and take $i \in [n-1]$ for some n. By arity-monotonicity and nondecreasingness we have $\mathsf{w}_n(x) \wedge c_{i,n} = \mathsf{M}_{\triangle,\mathbf{w}}(i * x, (n-i) * 0) = \mathsf{M}_{\triangle,\mathbf{w}}((i+1) * x, (n-i) * 0) = \mathsf{M}_{\triangle,\mathbf{w}}((i+1) * x, (n-i-1) * 0) = \mathsf{w}_n(x) \wedge c_{i+1,n} = \mathsf{w}_1(b)$, and the proof is complete. □

Again, note that each max-insensitive, nondecreasing and arity-monotonic symmetric aggregation operator is also zero-insensitive. We see that the only max-insensitive OM3 operators are nondecreasing functions of the Max operator.

The three following properties have a similar spirit as the ones above. They in turn guarantee that the output value will surely change if an element of some kind will be added to the input vector.

Firstly, the following implies that the aggregation operator is "production sensitive", i.e. an output of any new product always affect overall valuation.

Proposition 16. *Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 2. Then* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is* ***zero-sensitive****, i.e. such that for each* $\mathbf{x} \in \mathbb{I}^{0,1,\cdots}$ *it holds* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, 0) > \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$*, if and only if*

(i) $(\forall n)$ $(\forall x \in \mathbb{I})$ $\mathsf{w}_{n+1}(x) > \mathsf{w}_n(x) > 0$,
(ii) $(\forall n)$ $(\forall i \in [n])$ $c_{i,n+1} > c_{i,n}$.

Proof. ($\Longleftarrow$) Trivial.

($\Longrightarrow$) Fix n. Firstly, let $\mathbf{x} = (n * 0)$. By Lemma 2 we have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{w}_n(0) \wedge c_{n,n} = \mathsf{w}_n(0)$. On the other hand, $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, 0) = \mathsf{w}_{n+1}(0) \wedge c_{n,n} = \mathsf{w}_{n+1}(0)$, thus necessarily $\mathsf{w}_{n+1}(0) > \mathsf{w}_n(0)$ and, in particular for $n = 1$, $\mathsf{w}_1(0) > 0$.

Now take any $i \in [n]$ and let $\mathbf{x} = (i * b, (n - i) * 0)$. We have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = (\mathsf{w}_n(b) \wedge c_{i,n}) \vee (\mathsf{w}_n(0) \wedge c_{n,n}) = c_{i,n}$. On the other hand, $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, 0) = (\mathsf{w}_{n+1}(b) \wedge c_{i,n+1}) \vee (\mathsf{w}_{n+1}(0) \wedge c_{n+1,n+1}) = c_{i,n+1}$, thus $c_{i,n+1} > c_{i,n}$.

Moreover, consider any $y \in \mathbb{I}$ and let $\mathbf{x} = (n * y)$. We have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = (\mathsf{w}_n(y) \wedge c_{n,n}) = \mathsf{w}_n(y) < \mathsf{w}_{n+1}(y) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, 0)$, thus our sufficient conditions are also necessary, QED. □

Each zero-sensitive aggregation operator is of course arity-monotonic; in fact, this property may also be called strict arity-monotonicity.

The h-index and Max do not fulfill this property. However, e.g. $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = n\widetilde{\mathsf{H}}(\mathbf{x} + 1) = \bigvee_{i=1}^{n}(n\,(1 + x_{(n-i+1)})) \wedge (ni)$, $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = n + \widetilde{\mathsf{H}}(\mathbf{x}) = \bigvee_{i=1}^{n}(n + x_{(n-i+1)}) \wedge (n + i)$, or even $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = n$ do obey this condition.

$F+$sensitivity, see [21,50], cf. also "productivity responsiveness" in [39], states that if we add an element greater than current overall valuation, the output value will surely be affected. We discuss this property together with zero-insensitivity, otherwise the form of $\triangle$ and $\mathbf{w}$ becomes very complicated.

Proposition 17. *(See [9].) Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 2. Then* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is zero-insensitive and* $F+$***sensitive****, i.e. for all* $\mathbf{x} \in \mathbb{I}^{0,1,\cdots}$ *and* $y > \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$ *we have* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, y) > \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$*, if and only if there exist:*

(i) *a function* w *such that* $\mathsf{w}(x) \ge x$ *for all* x*, and strictly increasing for* $x : \mathsf{w}(x) < \mathsf{w}(b)$*,*
(ii) *a sequence* $(c_1, c_2, \ldots)$ *such that for* $c_i < \mathsf{w}(b)$ *we have* $\mathsf{w}(x) < c_i$ *for all* $x : \mathsf{w}(x) < \mathsf{w}(b)$ *and* $c_i < c_{i+1}$*,*

such that $\mathsf{w}_n = \mathsf{w} \wedge c_n$ *and* $c_{i,n} = c_i$*.*

See [9, Theorem 4] for the proof. It is easily seen that if $\mathsf{M}_{\triangle,\mathbf{w}}$ is continuous, then a zero-insensitive and $F+$sensitive OM3 operator is of the form $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{w}(\mathsf{Max}(\mathbf{x}))$.

On the other hand, each zero-sensitive (note again that the proposition is restricted to zero-insensitive functions only) aggregation operator is also $F+$sensitive.

The next property is also examined together with zero-insensitivity and continuity, for the same reasons as above.

Proposition 18. *Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 2. Then* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is zero-insensitive, continuous, and* ***max+sensitive****, i.e. for all* $\mathbf{x} \in \mathbb{I}^{0,1,\cdots}$ *and* $y > \mathsf{Max}(\mathbf{x})$ *we have* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, y) > \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$*, if and only if there exist* $(c_1, c_2, \ldots)$ *and* $\mathsf{w} : \mathbb{I} \to \mathbb{I}$ *fulfilling:*

(i) $c_1 = c_2 = \cdots = \mathsf{w}(b)$,
(ii) w *is strictly increasing,*

such that $\mathsf{w}_n = \mathsf{w} \wedge c_n$ *and* $c_{i,n} = c_i$*.*

Proof. First of all, it is easily seen that a zero-insensitive OM3 operator with $c_1 = \mathsf{w}(b)$ is max+sensitive if and only if w is strictly increasing. This is because in such case we have $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{w}(x_{(n)})$.

Let us show that the assumption $c_1 < \mathsf{w}(b)$ leads to a contradiction. Indeed, if we take any n and then some x such that $\mathsf{w}(x) \in [c_n, c_{n+1})$, then we have $\mathsf{M}_{\triangle,\mathbf{w}}((n+1) * x) = \mathsf{w}(x) \wedge c_{n+1} = \mathsf{w}(x) \not< \mathsf{M}_{\triangle,\mathbf{w}}(y, (n+1) * x) = \mathsf{w}(x) \wedge c_{n+2} = \mathsf{w}(x)$. □

We see that the only zero-insensitive, continuous and max+sensitive OM3 operators are strictly increasing functions of $\mathsf{Max}(\mathbf{x})$.

On the other hand, among discontinuous, zero-insensitive and max+sensitive OM3 operators we may find e.g. $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \bigvee_{i=1}^{n} (b\mathbf{I}(x_{(n-i+1)})) \wedge i$ (for $b = \infty$). Also, each zero-sensitive function is also max+sensitive.

3.3. Properties guaranteeing consistency of rankings

Another very important group of properties consists of arity-dependent conditions, but deserves to be treated separately. All of the properties discussed below share the same idea: given a tuple of producers, their relative ordering should not change if we improve their state in the same way. Here, we shall study only zero-insensitive aggregation operators, otherwise the form of $\triangle$ and $\mathbf{w}$ becomes quite complicated. Unfortunately, it will turn out that only very simple OM3 operators meet these conditions.

The following property is inspired by an illustration presented in [48]. It states that if two producers gain the same relative improvement in the products valuations (e.g. they become to be two times better), the way they are ranked should not change.

Proposition 19. *Let $b = \infty$, $\mathbf{w}$ and $\triangle$ be of the form given in Lemma 2. Then $\mathsf{M}_{\triangle,\mathbf{w}}$ is zero-insensitive and* ***multiplicative coherent****, i.e. for all $\mathbf{x}, \mathbf{y} \in \mathbb{I}^{0,1,\cdots}$ and $d \geq 1$ if $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \leq \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y})$, then $\mathsf{M}_{\triangle,\mathbf{w}}(d\mathbf{x}) \leq \mathsf{M}_{\triangle,\mathbf{w}}(d\mathbf{y})$, if and only if there exist:*

(i) *a nondecreasing function w,*
(ii) *a sequence $(c_1, c_2, \ldots)$ such that for all k it either holds $(\forall x > 0)\ c_k \leq \mathsf{w}(x)$ or $c_k = \mathsf{w}(b)$,*

such that $\mathsf{w}_n = \mathsf{w} \wedge c_n$ and $c_{i,n} = c_i$.

Proof. ($\Longrightarrow$) Note that this property implies that if $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y})$, then for all $d \geq 1$ $\mathsf{M}_{\triangle,\mathbf{w}}(d\mathbf{x}) = \mathsf{M}_{\triangle,\mathbf{w}}(d\mathbf{y})$.

Take any j, i such that $c_j < \mathsf{w}(b)$ and $c_i = \mathsf{w}(b)$. Moreover, take $x > 0$ and d (possibly such that $dx = b$) for which we have $\mathsf{w}(x) \leq c_j$ and $\mathsf{w}(dx) > c_j$. Thus, $\mathsf{M}_{\triangle,\mathbf{w}}(j * x) = \mathsf{M}_{\triangle,\mathbf{w}}(i * x)$. However, $c_j = \mathsf{M}_{\triangle,\mathbf{w}}(j * (dx))$ and $\mathsf{M}_{\triangle,\mathbf{w}}(i * (dx)) = \mathsf{w}(dx) > c_j$, a contradiction.

Thus, for any k it necessarily holds $c_k = \mathsf{w}(b)$ or $c_k \leq \mathsf{w}(x)$ for all $x > 0$.

($\Longleftarrow$) Take any $\mathbf{x}, \mathbf{y} \in \mathbb{I}^{0,1,\cdots}$ such that $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \leq \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y})$. As we deal with zero-insensitive aggregation operators here, with no loss in generality we may assume that $x_{(1)} > 0$ and $y_{(1)} > 0$. Let $n = |\mathbf{x}|$, $m = |\mathbf{y}|$, and $k = \min\{k : c_k = \mathsf{w}(b)\}$.

If $n, m < k$, then $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = c_n$ and $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y}) = c_m$. Thus, for all $d \geq 1$ we obviously have $\mathsf{M}_{\triangle,\mathbf{w}}(d\mathbf{x}) \leq \mathsf{M}_{\triangle,\mathbf{w}}(d\mathbf{y})$.

If $n, m \geq k$, then $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{w}(x_{(n-k+1)})$ and $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y}) = \mathsf{w}(y_{(m-k+1)})$. By nondecreasingness of w, again the property holds.

If $n < k$ and $m \geq k$, $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = c_n \leq \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y}) = \mathsf{w}(y_{(m-k+1)}) \leq \mathsf{w}(dy_{(m-k+1)}) = \mathsf{M}_{\triangle,\mathbf{w}}(d\mathbf{y})$.

As $n \geq k$ and $m < k$ is impossible, the proof is complete. □

Under continuity, only sample quantiles fulfill this property. Moreover, OM3 operators equivalent to the number of elements in a sequence or the number of non-zero elements also obey this property.

One may also require a proper behavior of aggregation operators with respect to a different type of products' valuations improvement.

Proposition 20. *Let $b = \infty$, $\mathbf{w}$ and $\triangle$ be of the form given in Lemma 2. Then $\mathsf{M}_{\triangle,\mathbf{w}}$ is zero-insensitive and* ***additive coherent****, i.e. for all $\mathbf{x}, \mathbf{y} \in \mathbb{I}^{0,1,\cdots}$ and $e \geq 0$ if $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \leq \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y})$, then $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x} + e) \leq \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y} + e)$, if and only if $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) = \mathsf{w}(\mathsf{Max}(\mathbf{x}))$ for some nondecreasing w.*

Proof. ($\Longrightarrow$) As we consider a zero-insensitive OM3 operator, assume that it is generated by a function w and a sequence $(c_1, c_2, \ldots)$. We have $\mathsf{w}(0) \leq c_1$. Moreover, $\mathsf{M}_{\triangle,\mathbf{w}}(0) = \mathsf{w}(0) = \mathsf{M}_{\triangle,\mathbf{w}}(n * 0)$ for all $n \geq 1$. If $c_1 < \mathsf{w}(b)$, then there exists $j > 1$ and $e > 0$ such that $\mathsf{M}_{\triangle,\mathbf{w}}(e) = c_i \neq \mathsf{M}_{\triangle,\mathbf{w}}(j * e)$. Thus, $\mathsf{M}_{\triangle,\mathbf{w}}$ is necessarily such that $c_1 = c_2 = \cdots$.

($\Longleftarrow$) Trivial. □

The “independence” property, which was considered in [7], states that the relative ranking of two producers should not change after an addition of products of the same quality.

Proposition 21. *Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 2. Then* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is zero-insensitive and* ***independent****, i.e. for all* $\mathbf{x}, \mathbf{y} \in \mathbb{I}^{0,1,\dots}$ *and* $z \in \mathbb{I}$ *it holds* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \le \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y}) \Rightarrow \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, z) \le \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y}, z)$*, if and only if there exist:*

(i) *a sequence* $(c_1, c_2, \dots)$ *with* $c_i = c_{i+1} = \cdots$ *and* $c_k < c_i$ *for some* i *and* $k < i$*,*
(ii) *a nondecreasing function* w *such that for any* x *either* $\mathsf{w}(x) < c_1$ *or* $\mathsf{w}(x) = \mathsf{w}(b)$*,*

such that $\mathsf{w}_n = \mathsf{w} \wedge c_n$ *and* $c_{i,n} = c_i$*.*

Proof. ($\Longleftarrow$) Take any $n, m, \mathbf{x} \in \mathbb{I}^n$, $\mathbf{y} \in \mathbb{I}^m$ fulfilling $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \le \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y})$. Additionally, let $z \in \mathbb{I}$. $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$ is equal to $\mathsf{w}(x_{(n)})$ or c_i for some $i \in [n]$. $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y})$ in turn is equal to $\mathsf{w}(y_{(m)})$ or c_j for some $j \in [m]$. By examining all the possible cases we easily get that $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, z) \le \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y}, z)$.

($\Longrightarrow$) Let w and $(c_1, c_2, \dots)$ be such that $\mathsf{w}_n = \mathsf{w} \wedge c_n$ and $c_{i,n} = c_i$.

Take any x such that $\mathsf{w}(x) \ge c_1$. We have $\mathsf{M}_{\triangle,\mathbf{w}}(x) = \mathsf{M}_{\triangle,\mathbf{w}}(b) = c_1$. Our property implies that $\mathsf{M}_{\triangle,\mathbf{w}}(b, x) = c_1 \vee \mathsf{w}(x) \wedge c_2 = \mathsf{M}_{\triangle,\mathbf{w}}(b, b) = \mathsf{w}(b) \wedge c_2 = c_2$. Consequently, we obtain that $\mathsf{w}(x) \ge c_n$ for all n. Therefore, $\mathsf{w}(x) = \mathsf{w}(b)$.

Now assume that there exists i such that $c_i = c_{i+1}$. Then $\mathsf{M}_{\triangle,\mathbf{w}}(i * b) = c_i = \mathsf{M}_{\triangle,\mathbf{w}}((i+1) * b)$ and consequently our property implies that $c_i = c_n$ for all $n \ge i$. Thus, $c_i = \mathsf{w}(b)$. □

Note that under continuity this property holds iff an OM3 operator is a nondecreasing function of Max. Moreover, e.g. the OM3 operator equivalent to the number of nonzero elements in a sample also fulfills this property.

Here is another property from [7]. It considers joint output of consortia of producers: if a producer A is dominated by producer B, and C is dominated by D, then it is reasonable that A and C together (i.e. their concatenated outputs) shall be dominated by B and D.

Proposition 22. *Let* $\mathbf{w}$ *and* $\triangle$ *be of the form given in Lemma 2. Then* $\mathsf{M}_{\triangle,\mathbf{w}}$ *is zero-insensitive and* ***consistent****, i.e. for all* $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}'$ *such that* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \le \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y})$ *and* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}') \le \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y}')$ *it holds* $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}, \mathbf{x}') \le \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y}, \mathbf{y}')$*, if and only if there exist:*

(i) *a sequence* $(c_1, c_2, \dots)$*,*
(ii) *a nondecreasing function* w *such that for any* x *either* $\mathsf{w}(x) < c_1$ *or* $\mathsf{w}(x) = \mathsf{w}(b)$*,*

such that $\mathsf{w}_n = \mathsf{w} \wedge c_n$ *and* $c_{i,n} = c_i$*.*

Proof. ($\Longleftarrow$) Assume there exists $x \ne b$ such that $\mathsf{w}(x) \ge c_1$. Then $\mathsf{M}_{\triangle,\mathbf{w}}(x) = c_1 = \mathsf{M}_{\triangle,\mathbf{w}}(b)$. Then, for any n it should hold $\mathsf{M}_{\triangle,\mathbf{w}}(n * x) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})(n * b) = c_n$. Thus, $\mathsf{w}(x) = \mathsf{w}(b)$.

($\Longrightarrow$) Take any $\mathbf{x}' \in \mathbb{I}^{n'}$, $\mathbf{x}'' \in \mathbb{I}^{n''}$, $\mathbf{y}' \in \mathbb{I}^{m'}$, $\mathbf{y}'' \in \mathbb{I}^{m''}$ such that $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}') \le \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y}')$ and $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}'') \le \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y}'')$.

Let $d' = \#\{x'_i : \mathsf{w}(x'_i) = \mathsf{w}(b)\}$, $d'' = \#\{x''_i : \mathsf{w}(x''_i) = \mathsf{w}(b)\}$, $e' = \#\{y'_i : \mathsf{w}(y'_i) = \mathsf{w}(b)\}$, and $e'' = \#\{y''_i : \mathsf{w}(y''_i) = \mathsf{w}(b)\}$. If $d' = 0$, then $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}') = \mathsf{w}(x'_{(n')})$ and $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}') = c_{d'}$ otherwise; and similarly for other vectors.

We of course have $d' \le e'$ and $d'' \le e''$. If $e' = e'' = 0$, then the property obviously holds, as $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}', \mathbf{x}'') = \mathsf{w}(x'_{(n')}) \vee \mathsf{w}(x''_{(n'')}) \le \mathsf{w}(y'_{(m')}) \vee \mathsf{w}(y''_{(m'')}) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y}', \mathbf{y}'')$.

Otherwise, denote with $\mathbf{x}''' := (\mathbf{x}', \mathbf{x}'')$, $\mathbf{y}''' := (\mathbf{y}', \mathbf{y}'')$, and let $d''' := d' + d''$, $e''' := e' + e''$. If $d''' = 0$, then $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}''') = \mathsf{w}(x_{(n'+n'')})$, and $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}''') = c_{d'''}$ otherwise. In both cases these values are not greater than $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{y}''') = c_{e'''}$, QED. □

Yet again, under continuity, a consistent zero-insensitive OM3 operator is equivalent to a nondecreasing function of Max.

3.4. Properties providing output value normalization

The last group of the properties discussed concerns the normalization of an aggregation operator's output value. Although these conditions are not important in ranking problems, they may be useful when one is trying to *measure* a producer's performance. Generally, the following properties try to "calibrate" the output value according to some vector forms, so that one producer may be said that he/she is γ times better than another. Although proving most of them for OM3 is very easy, they seem to be quite important for practitioners and thus they are worth to be stated explicitly.

In the classical approach to aggregation, one of the most widely discussed property is the idempotence, see [29], which holds for a given aggregation operator F iff for all $x \in \mathbb{I}$ and $n \in \mathbb{N}$ we have $\mathsf{F}(n * x) = x$. We may note that the only idempotent OM3 operator is equivalent to Max. However, in an arity-dependent setting such a condition may not seem very important, as in PAP performance/impact measures are needed, and not measures of data central tendency.

The notion of idempotency is generalized by its asymptotic version, cf. [25]. It is easily seen that:

Proposition 23. *An OM3 operator $\mathsf{M}_{\triangle,\mathbf{w}}$ is* ***asymptotically idempotent****, i.e. for any $x \in \mathbb{I}$ it holds $\lim_{n\to\infty} \mathsf{M}_{\triangle,\mathbf{w}}(n * x) = x$, if and only if $\lim_{n\to\infty} \mathsf{w}_n(x) = x$ and $\lim_{n\to\infty} c_{n,n} = b$.*

The Max operator as well as the generalized Hirsch index $\widetilde{\mathsf{H}}$ (with $\mathsf{w}_n(x) = x$, $c_{i,n} = i$) are examples of asymptotic OM3 operators.

The following property may be conceived as an arity-sensitive version of idempotency for PAP. It states that n products "worth" n units each shall be "worth" n units overall.

Proposition 24. *Let $\mathbf{w}$ and $\triangle$ be of the form given in Lemma 2. Then $\mathsf{M}_{\triangle,\mathbf{w}}$ for all n fulfills $\mathsf{F}(n * n) = n$, if and only if $c_{n,n} \geq n$ and $\mathsf{w}_n(n) = n$.*

The proof is omitted. Note that $\mathsf{w}_n(n) = n$ implies $c_{n,n} \geq n$.

Note that the property itself is not arity-dependent. Among OM3 operators fulfilling it we have e.g. the h-index, Max and $\widetilde{\mathsf{H}}$. Moreover, idempotence implies this property.

On the other hand, one may require that n best possible products are "worth" exactly n.

Proposition 25. *Let $\mathbf{w}$ and $\triangle$ be of the form given in Lemma 2. Then $\mathsf{M}_{\triangle,\mathbf{w}}$ for all n fulfills $\mathsf{F}(n * b) = n$ if and only if $c_{n,n} = n$.*

The proof is omitted. This property is also not arity-dependent and is fulfilled by e.g. the h-index.

Other properties of this kind may require e.g. that $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \leq \mathsf{Max}(\mathbf{x})$ (which is fulfilled iff $\mathsf{w}_n(x) \leq x$) or that $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \leq n$ (met iff $c_{n,n} \leq n$).

Note that although $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) \geq \mathsf{Min}(\mathbf{x})$ iff $\mathsf{w}_n(x) \geq x$ and $c_{n,n} = \mathsf{w}_n(b)$, this condition together with arity-monotonicity leads to OM3 operators equivalent to some function of Max. Thus, such a condition seems to be too strong for PAP and, as far as the OM3 class is concerned, we would rather restrict the output value from above (w.r.t. quality or quantity of elements).

What is more, one may sometimes require e.g. sub-homogeneity degree 1 (for all $n \in \mathbb{N}$, $d \geq 1$, and $\mathbf{x} \in \mathbb{I}^n$ it holds $\mathsf{M}_{\triangle,\mathbf{w}}(d\mathbf{x}) \leq d\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$), which is fulfilled iff $\mathsf{w}_n(dx) \leq d\mathsf{w}_n(x)$, or sub-additivity (for all $n \in \mathbb{N}$, $e \geq 0$, and $\mathbf{x} \in \mathbb{I}^n$ it holds $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x} + e) \leq \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) + e$) which is obtained iff $\mathsf{w}(x + e) \leq \mathsf{w}_n(x) + e$, etc.

Note that the only OM3 operator fulfilling homogeneity $\mathsf{M}_{\triangle,\mathbf{w}}(d\mathbf{x}) = d\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x})$ or additivity $\mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x} + e) = \mathsf{M}_{\triangle,\mathbf{w}}(\mathbf{x}) + e$ is the Max function.

Table 2 summarizes the properties investigated in this section.

Let us consider the following exemplary OM3 operators:

1. $\mathsf{Max}(\mathbf{x}) = x_{(n)}$ (sample maximum),
2. $\mathsf{Max2}(\mathbf{x}) = x_{(n)}^2$,
3. $\mathsf{MaxN}(\mathbf{x}) = x_{(n)} \wedge n$,
4. $\mathsf{Q5}(\mathbf{x}) = x_{(n-5+1)}$ if $n \geq 5$ and 0 otherwise ($\sim$ fifth quantile, arity-monotonic),

Table 2
Major properties studied.

Property	Name	Result for OM3
$\mathsf{F}(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) = \mathsf{F}(\mathbf{x}) \wedge \mathsf{F}(\mathbf{y})$	symmetric minitivity	see [18]
$\mathsf{F}(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) = \mathsf{F}(\mathbf{x}) \wedge \mathsf{F}(\mathbf{y})$	symmetric maxitivity	see [18]
$\mathsf{F}(\mathbf{x} \overset{S}{\wedge} \mathbf{y}) + \mathsf{F}(\mathbf{x} \overset{S}{\vee} \mathbf{y}) = \mathsf{F}(\mathbf{x}) + \mathsf{F}(\mathbf{y})$	symmetric modularity	see [18]
$\mathbf{x} \le \mathbf{y} \Rightarrow \mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{y})$	nondecreasingness	Lemma 2
$\mathbf{x} <_1 \mathbf{y} \Rightarrow \mathsf{F}(\mathbf{x}) < \mathsf{F}(\mathbf{y})$	sensitivity	Proposition 9
$\mathbf{x} < \mathbf{y} \Rightarrow \mathsf{F}(\mathbf{x}) < \mathsf{F}(\mathbf{y})$	weak sensitivity	Proposition 10
$\lim_{\mathbf{x}\to\mathbf{x}_0} \mathsf{F}(\mathbf{x}) = \mathsf{F}(\mathbf{x}_0)$	continuity	Proposition 11
$\mathsf{F}(\mathbf{x}, 0) \ge \mathsf{F}(\mathbf{x})$	arity-monotonicity	Proposition 12
$\mathsf{F}(\mathbf{x}, 0) = \mathsf{F}(\mathbf{x})$	zero-insensitivity	Proposition 13
$\mathsf{F}(\mathbf{x}, \mathsf{F}(\mathbf{x})) = \mathsf{F}(\mathbf{x})$	F-insensitivity	Proposition 14
$\mathsf{F}(\mathbf{x}, \mathsf{Max}(\mathbf{x})) = \mathsf{F}(\mathbf{x})$	max-insensitivity	Proposition 15
$\mathsf{F}(\mathbf{x}, 0) > \mathsf{F}(\mathbf{x})$	zero-sensitivity	Proposition 16
$\mathsf{F}(\mathbf{x}, \mathsf{F}(\mathbf{x}) + \varepsilon) > \mathsf{F}(\mathbf{x}), \varepsilon > 0$	F+sensitivity	Proposition 17
$\mathsf{F}(\mathbf{x}, \mathsf{Max}(\mathbf{x}) + \varepsilon) > \mathsf{F}(\mathbf{x}), \varepsilon > 0$	max+sensitivity	Proposition 18
$\mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{y}) \Rightarrow \mathsf{F}(d\mathbf{x}) \le \mathsf{F}(d\mathbf{y}), d \ge 1$	multiplicative coherent	Proposition 19
$\mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{y}) \Rightarrow \mathsf{F}(\mathbf{x} + e) \le \mathsf{F}(\mathbf{y} + e), e \ge 0$	additive coherent	Proposition 20
$\mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{y}) \Rightarrow \mathsf{F}(\mathbf{x}, z) \le \mathsf{F}(\mathbf{y}, z), z \in \mathbb{I}$	independence	Proposition 21
$\mathsf{F}(\mathbf{x}) \le \mathsf{F}(\mathbf{y})$ and $\mathsf{F}(\mathbf{x}') \le \mathsf{F}(\mathbf{y}') \Rightarrow \mathsf{F}(\mathbf{x}, \mathbf{x}') \le \mathsf{F}(\mathbf{y}, \mathbf{y}')$	consistency	Proposition 22
$\lim_{n=\infty} \mathsf{F}(n * x) = x$	asymptotic idempotence	Proposition 23
$\mathsf{F}(n * n) = n$	n times n equals n	Proposition 24
$\mathsf{F}(n * b) = n$	n times b equals n	Proposition 25

5. $\mathsf{Q15}(\mathbf{x}) = x_{(n-(5\wedge n)+1)}$ ($\sim$ fifth quantile, not arity-monotonic),
6. $\mathsf{H}(\mathbf{x}) = \bigvee_{i=1}^{n} \lfloor x_{(n-i+1)} \rfloor \wedge i$ (the Hirsch index),
7. $\widetilde{\mathsf{H}}(\mathbf{x}) = \bigvee_{i=1}^{n} x_{(n-i+1)} \wedge i$ (a generalized Hirsch index),
8. $\mathsf{H2}(\mathbf{x}) = \bigvee_{i=1}^{n} \lfloor \sqrt{x_{(n-i+1)}} \rfloor \wedge i$ (the $h^{(2)}$-index [36]),
9. $\widetilde{\mathsf{H2}}(\mathbf{x}) = \bigvee_{i=1}^{n} \sqrt{x_{(n-i+1)}} \wedge i$ (a generalized $h^{(2)}$ index),
10. $\mathsf{N}(\mathbf{x}) = n$ (sample length),
11. $\mathsf{NP}(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{I}(x_i > 0) = \bigvee_{i=1}^{n} \mathbf{I}(x_{(n-i+1)} > 0)b \wedge i$ (number of elements with non-zero quality).

Table 3 summarizes which of the properties discussed in this section are fulfilled by the above functions. Quite surprisingly, the OM3 operator that obeys the greatest number of properties is the Max function. Of course, recall that all the functions additionally fulfill symmetry, nondecreasingness, symmetric maxitivity, minitivity, and modularity.

4. Probabilistic properties of OM3 operators

In this section the most fundamental probabilistic properties of OM3 operators are considered. Special attention is paid to the probability distribution of this class of functions and to the behavior of its basic characteristics, like expected value and variance, in an arity-dependent setting.

First of all, we should note that in [22], basic probabilistic and statistical properties of the so-called S-statistics were studied. This class of functions is a particular subclass of OM3 operators. However, the study concerned the asymptotic behavior of functions under the assumption that they are averaging aggregation operators, cf. [29], and these results cannot be straightforwardly extrapolated to our framework.

Throughout this section we will restrict ourselves only to OM3 operators fulfilling the zero-insensitivity property. On account of Theorem 13, such operators may be described by a nondecreasing function w and a sequence $(c_1, c_2, \ldots)$ such that $(\forall n)\ \mathsf{w}_n = \mathsf{w} \wedge c_n$ and $c_{i,n} = c_{i,n+1}$. It implies that the form of w does not change as n increases. Therefore, for the purpose of investigating e.g. the asymptotic behavior of the expected value, this assumption is necessary.

In order to characterize the distribution of OM3 operators, the following lemma will be useful.

Table 3
Exemplary OM3 operators and properties that they fulfill.

Property name	Max	Max2	MaxN	Q5	Q15	H	$\widetilde{\mathsf{H}}$	H2	$\widetilde{\mathsf{H2}}$	N	NP	$\sum$
arity-monotonicity	●	●	●	●	○	●	●	●	●	●	●	10
sensitivity	○	○	○	○	○	○	○	○	○	○	○	0
weak sensitivity	●	●	○	○	●	○	○	○	○	○	○	3
continuity	●	●	●	●	●	○	●	○	●	●	○	8
zero-insensitivity	●	●	○	●	○	●	●	●	●	○	●	8
F-insensitivity	●	○	○	●	○[a]	●	●	●	○	○	○	5
max-insensitivity	●	●	○	○	○[a]	○	○	○	○	○	○	2
zero-sensitivity	○	○	○	○	○	○	○	○	○	●	○	1
F+sensitivity	●	●	●[a]	○	○[a]	○	○	○	○	●[a]	●	5
max+sensitivity	●	●	●[a]	○	○[a]	○	○	○	○	●[a]	●[a]	5
multiplicative coherent	●	●	○[a]	●	●[a]	○	○	○	○	●[a]	●	6
additive coherent	●	●	○[a]	○	●[a]	○	○	○	○	●[a]	○	4
independence	●	●	○[a]	○	○[a]	○	○	○	○	●[a]	●	4
consistency	●	●	●[a]	○	○[a]	○	○	○	○	●[a]	●	5
asymptotic idempotence	●	○	●	●	●	○	●	○	○	○	○	5
n times n equals n	●	○	●	○	●	●	●	○	○	●	●	7
n times b equals n	○	○	●	○	○	●	●	●	●	●	●	7
$\sum$	14	11	8	6	7	5	7	4	4	11	9	$\sum$

[a] Denotes cases proven separately.

Lemma 26. *Let us fix n. It holds* $\widetilde{\mathsf{H}}(\mathbf{x}) = \bigvee_{i=1}^{n} x_{(n-i+1)} \wedge i = \max\{x : \sum_{i=1}^{n} \mathbf{I}(x_i \geqslant x) \geqslant x\}$, *where* $\mathbf{I}(S) = 1$ *iff S is true, and* 0 *otherwise.*

Proof. We have

$$\begin{aligned}\widetilde{\mathsf{H}}(\mathbf{x}) &= \max\{x_{(n-i+1)} : x_{(n-i+1)} \leqslant i\} \vee \max\{i : x_{(n-i+1)} \geqslant i\} \\ &= \max\Big\{\big\{x : \max\{x_{(n-i+1)} : x_{(n-i+1)} \leqslant i\} \geqslant x\big\} \cup \big\{x : \max\{i : x_{(n-i+1)} \geqslant i\} \geqslant x\big\}\Big\} \\ &= \max\big\{x : \max\{i : x_{(n-i+1)} \geqslant x\} \geqslant x\big\}.\end{aligned}$$

What is more, it is easily seen that $\max\{i : x_{(n-i+1)} \geqslant x\} = \sum_{i=1}^{n} \mathbf{I}(x_i \geqslant x)$. Thus, $\widetilde{\mathsf{H}}(\mathbf{x}) = \max\{x : \sum_{i=1}^{n} \mathbf{I}(x_i \geqslant x) \geqslant x\}$, and the proof is complete. □

Let $(X_1, \ldots, X_n)$ denote a sample of independent, identically distributed (i.i.d.) random variables with a common cumulative distribution function (c.d.f.) F defined on $\mathbb{I} = [0, b]$, possibility with $b = \infty$, i.e. supp $F \subseteq \mathbb{I}$.

In the context of PAP, products of high quality ought to appear not very often. For example, in scientometric modeling F may be assumed to be a heavy-tailed, right-skewed distribution, like Pareto-type II (defined as $F(x) = 1 - (1 + x/s)^{-k}$, $x \in [0, \infty]$, $k, s > 0$) or exponential distribution (with $F(x) = 1 - \exp(-\lambda x)$, $x \in [0, \infty]$, $\lambda > 0$), cf. e.g. [3,27,28]. Therefore, the assumption that $\mathbb{I} = [0, b]$, with possibly $b = \infty$, appears quite naturally.

Theorem 27. *Fix n. Let* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *i.i.d. G and* $M_n = \widetilde{\mathsf{H}}(\mathbf{Y}) = \bigvee_{i=1}^{n} Y_{(n-i+1)} \wedge i$. *Then the cumulative distribution function of* M_n *is given by*

$$\Pr(M_n \leqslant x) = \mathcal{I}\big(G(x); n - \lfloor x \rfloor, \lfloor x \rfloor + 1\big),$$

where $\mathcal{I}$ *denotes the regularized incomplete Beta function, i.e.* $\mathcal{I}(x; a, b) = \frac{1}{\mathcal{B}(a,b)} \int_0^x t^{a-1}(1-t)^{b-1}\,dt$, $a, b > 0$ *and* $x \in [0, 1]$.

Proof. Our proof starts with the observation that according to Lemma 26, we have $M_n = \max\{x : \sum_{i=1}^{n} \mathbf{I}(Y_i \geqslant x) \geqslant x\}$. Thus, it holds $M_n = \max\{x : Y_{(n-\lceil x \rceil+1)} \geqslant x\}$. Let us show that $\Pr(M_n > y) = \Pr(Y_{(n-\lfloor y \rfloor)} > y)$.

Please note that $M_n > y$ implies $n - \lceil M_n \rceil + 1 \leqslant n - \lfloor y \rfloor$ and $Y_{(n-\lfloor y \rfloor)} \geqslant Y_{(n-\lceil M_n \rceil+1)} > y$. As $M_n > y \Rightarrow Y_{(n-\lfloor y \rfloor)} > y$, we have $\Pr(Y_{(n-\lfloor y \rfloor)} > y \mid M_n > y) = 1$.

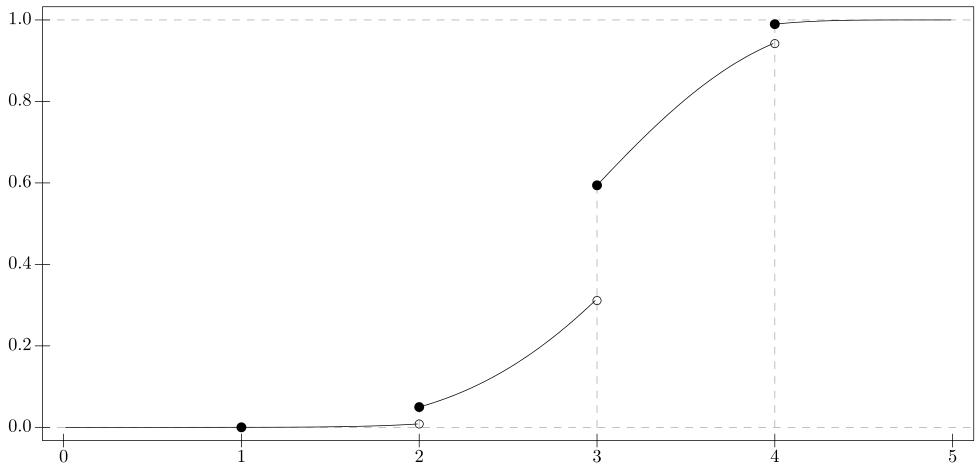

Fig. 2. Cumulative distribution function of an OM3 operator $\mathsf{M}_{\triangle,\mathbf{w}} = \bigvee_{i=1}^{n} Y_{(n-i+1)} \wedge i$ for an i.i.d. random sample $(Y_1, \ldots, Y_n)$ following the uniform distribution on [0, 5] and $n = 8$.

Now we will show that $Y_{(n-\lfloor y \rfloor)} > y \Rightarrow M_n > y$. It is clear to see that $(\forall y)$ we have $n - \lfloor y \rfloor \leqslant n - \lceil y \rceil + 1$. Thus, $y < Y_{(n-\lfloor y \rfloor)} \leqslant Y_{(n-\lceil y \rceil+1)}$. This implies $y \in \{x : Y_{(n-\lceil x \rceil+1)} \geqslant x\}$ and $y < M_n = \max\{x : Y_{(n-\lceil x \rceil+1)} \geqslant x\}$. Since $Y_{(n-\lfloor y \rfloor)} > y \Rightarrow M_n > y$, we have $\Pr(M_n > y \mid Y_{(n-\lfloor y \rfloor)} > y) = 1$. Therefore, $\Pr(M_n > x) = \Pr(Y_{(n-\lfloor x \rfloor)} > x)$.

It is well known that the distribution of the i-th order statistic from an i.i.d. sample is of the form $Y_{(i)} \sim \mathcal{I}(G(x); i, n-i+1)$, see e.g. [11]. As a consequence, we get:

$$\Pr(M_n > x) = 1 - \mathcal{I}\big(G(x); n - \lfloor x \rfloor, \lfloor x \rfloor + 1\big),$$

and the proof is complete. □

Remark 28. For any function $\mathsf{w} : \mathbb{I} \to \mathbb{I}$, $(\mathsf{w}(X_1), \ldots, \mathsf{w}(X_n))$ is also a sample of i.i.d. random variables. It is well known that if X_i is a continuous random variable with a density f, then the distribution of $Y_i = \mathsf{w}(X_i)$ may be simply derived as $G(y) = \Pr(Y_i \leqslant y) = \Pr(\mathsf{w}(X_i) \leqslant y) = \int_{y:\mathsf{w}(x) \leq y} f(x)\, dx$. It is worth noting that if w is strictly increasing, then Y_i follows the distribution with c.d.f. $G := F \circ \mathsf{w}^{-1}$. However, generally even if X_i is continuous, Y_i need not be continuous. On the other hand, if X_i is a discrete random variable with a probability mass function $f(x) = \Pr(X_i = x)$, then the p.m.f. of Y_i is given by $g(y) = \Pr(\mathsf{w}(X_i) = y) = \sum_{y:\mathsf{w}(x)=y} f(y)$. □

It is worth pointing out that $\mathcal{I}(G(x); n - \lfloor x \rfloor, \lfloor x \rfloor + 1)$ may have discontinuities e.g. at points in $[n]$, even if G is continuous. Fig. 2 presents an exemplary graph of the c.d.f. of an OM3 operator.

Remark 29. On account of Lemma 6 and notation used therein, if an OM3 operator is such that the triangle of coefficients is of the form $c_1 < c_2 < \cdots < c_n$ for some n, then $M_{\triangle,\mathsf{w}}(\mathbf{x}) = \mathsf{f}_n(\bigvee_{i=1}^{n} \mathsf{w}'_n(x_{(n-i+1)}) \wedge i)$ for some increasing function f_n and nondecreasing function w'_n. Thus, the distribution of such an OM3 operator is defined as $\mathcal{I} \circ \mathsf{f}_n^{-1}$ and G' equal in case of i.a. increasing w_n to $G \circ \mathsf{w}_n'^{-1}$. □

Remark 30. Let us fix n. The h-index is given by $\mathsf{H}(x_1, \ldots, x_n) = \bigvee_{i=1}^{n} \lfloor x_{(n-i+1)} \rfloor \wedge i$. According to Lemma 6 and the notation used therein, we have $\mathsf{w}'_n(x) = \lfloor x \rfloor$ and $\mathsf{f}_n(x) = x$. Moreover, for an i.i.d. sample of random variables $(X_1, \ldots, X_n)$ with c.d.f. F, the distribution of $Y_i = \lfloor X_i \rfloor$ is given by $G(y) = \Pr(\lfloor X_i \rfloor \leqslant y) = \sum_{i=0}^{\lfloor y \rfloor}(F(i+1) - F(i)) = F(\lfloor y+1 \rfloor)$. Therefore, on account of Theorem 27, $\mathsf{H}(X_1, \ldots, X_n) \sim \mathcal{I}(F(\lfloor y+1 \rfloor), n - \lfloor y \rfloor, \lfloor y \rfloor + 1)$. Please note that this result is consistent with the one obtained in [20]. □

Taking the above remarks into account, from now on – with no loss in generality – we will consider only OM3 operators given by

$$\widetilde{\mathsf{H}}(Y_1, \ldots, Y_n) = \bigvee_{i=1}^{n} Y_{(n-i+1)} \wedge i,$$

where an i.i.d. random sample $(Y_1, \ldots, Y_n)$ follows some distribution G.

Note that the regularized incomplete beta function is the c.d.f. of a beta distribution and is also connected to the binomial distribution. Nevertheless, neither in a beta nor a binomial distribution, the second and the third parameters of $\mathcal{I}(x; a, b)$ do not depend on x. In our case, however, this nice property is not fulfilled, so any analytic calculations are much more difficult. Let us now recall some representations and properties of the regularized incomplete beta function.

Remark 31. For any integer a, it holds

$$1 - \mathcal{I}(x; a, b) = \mathcal{I}(1 - x; a, b) = (1 - x)^{a+b-1} \sum_{i=0}^{a-1} \binom{a+b-1}{i} \left(\frac{x}{1-x}\right)^{i},$$

see [1, Eq. 26.5.7], and

$$\mathcal{I}(x; a, b) = \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} x^{a}(1-x)^{b} + \mathcal{I}(1; a+1, b),$$

see [1, Eq. 26.5.16]. □

Remark 32. The expected value and variance of M_n are given, respectively, by

$$\mathbb{E}(M_n) = \int_{0}^{\infty} \left(1 - \mathcal{I}\big(G(x); n - \lfloor x \rfloor, \lfloor x \rfloor + 1\big)\right) dx,$$

$$\mathrm{Var}(M_n) = 2 \int_{0}^{\infty} x\left(1 - \mathcal{I}\big(G(x); n - \lfloor x \rfloor, \lfloor x \rfloor + 1\big)\right) dx - \big(\mathbb{E}(M_n)\big)^{2}. \qquad \square$$

Let us investigate the behavior of the expected value of OM3 operators. According to Remark 31, the expected value of M_n is given by

$$\mathbb{E}(M_n) = \int_{0}^{n} \left(\big(1 - G(x)\big)^{n} \sum_{i=0}^{n-\lfloor x \rfloor - 1} \binom{n}{i} \left(\frac{G(x)}{1 - G(x)}\right)^{i}\right) dx$$

We will show that the expected value of OM3 operators tends to infinity as $n \to \infty$. Before that we shall state the following auxiliary result.

Lemma 33. *Let* $(Y_1, Y_2, \ldots)$ *i.i.d.* G *such that* $\operatorname{supp} G = [0, \infty)$. *For fixed* i *let* $(Z_n^i)_{n \in \mathbb{N}, n \geq i}$ *be a sequence of random variables given by*

$$Z_n^i = Y_{(n-i+1)} \wedge i,$$

where $Y_{(n-i+1)} \sim \mathcal{I}(G(x); n - i + 1, i)$ *and* $Y_i \sim G$. *Then* Z_n^i *converges in distribution (weakly; see [45, Sec. 1.2.4]) to a random variable* Z^i *such that*

$$\Pr\big(Z^i = i\big) = 1.$$

Proof. Let $\mathbb{I} = [0, \infty]$. The proof starts with an observation that the distribution of Z_n^i is given by

$$F_n^i(x) = \begin{cases} \mathcal{I}(G(x); n - i + 1, i) & \text{if } x < i, \\ 1 & \text{if } x \geq i. \end{cases}$$

Let $x < i$. Please note that on account of Remark 31 we have:

$$F_n^i(x) := 1 - \mathcal{I}\big(1 - G(x); i, n-i+1\big) = G(x)^n \sum_{j=0}^{i-1} \binom{n}{j} \left(\frac{1-G(x)}{G(x)}\right)^j .$$

Moreover, if $(1 - G(x))/G(x) \leqslant 1$, then $F_n^i(x) \leqslant G(x)^n \sum_{j=0}^{i-1} \binom{n}{j} \leqslant G(x)^n (n+1)^{i-1}$. On the other hand, for $(1 - G(x))/G(x) > 1$ we have $F_n^i(x) \leqslant G(x)^n ((1-G(x))/G(x))^{i-1} \sum_{j=0}^{i-1} \binom{n}{j} \leqslant G(x)^n ((1-G(x))/G(x))^{i-1}(n+1)^{i-1}$. Hence, in both cases it holds $F_n^i(x) \leqslant C_i(x) G(x)^n (n+1)^{i-1}$, where $C_i(x) := ((1-G(x))/G(x))^{i-1} \geqslant 1$.

It may be easily shown that, by recursively applying the l'Hôpital rule, $a_n(x) = C_i(x)G(x)^n(n+1)^{i-1} \to 0$. Thus, as $0 \leqslant F_n^i(x) \leqslant a_n(x)$, on account of the squeeze theorem we have $\lim_{n\to\infty} F_n^i(x) = 0$ for $x < i$. On the other hand, obviously for $x \geq i$ and all n we have $F_n^i(x) = 1$.

Consequently, since $\lim_{n\to\infty} F_n^i(x) = F^i(x) = \begin{cases} 0 \text{ if } x<i \\ 1 \text{ if } x \geq i \end{cases}$, we have that $Z_n^i \xrightarrow{d} Z^i$, where Z_n^i follows a distribution $F^i(x)$, and therefore $\Pr(Z^i = i) = 1$. □

We can now formulate the main result concerning the limiting behavior of the expected value.

Theorem 34. *Let* $(Y_1, Y_2, \ldots)$ *i.i.d.* G *such that* $\operatorname{supp} G = [0, \infty)$. *Then,* $\mathbb{E}(M_n) \xrightarrow{n\to\infty} \infty$.

Proof. Note that $(\forall n \in \mathbb{N})$ $(\forall i \in [n])$ $M_n \geqslant Y_{(n-i+1)} \wedge i$. Therefore, M_n stochastically dominates Z_n^i almost everywhere, as $\Pr(M_n < Z_n^i) = 0$. It implies that $M_n \succeq_{FSD} Z_n^i$, where $\succeq_{FSD}$ denotes first order stochastic dominance. Hence, $\mathbb{E}(M_n) \geqslant \mathbb{E}(Z_n^i)$. It can be shown that:

$$\mathbb{E}(M_{n+1}) - \mathbb{E}(M_n) = \int_0^n \frac{\Gamma(n+1)}{\Gamma(n - \lfloor x \rfloor + 1)\Gamma(\lfloor x \rfloor + 1)} G(x)^{n-\lfloor x \rfloor} \big(1 - G(x)\big)^{\lfloor x \rfloor + 1}\, dx$$
$$+ \int_n^{n+1} \big(1 - \mathcal{I}\big(G(x); n - \lfloor x \rfloor + 1, \lfloor x \rfloor + 1\big)\big)\, dx \geqslant 0,$$

hence $\mathbb{E}(M_n)$ is a nondecreasing sequence. Let us denote $m := \lim_{n\to\infty} \mathbb{E}(M_n)$. Passing to the limit with $\mathbb{E}(M_n) \geqslant \mathbb{E}(Z_n^i)$, we obtain $m > i$. Since the above inequality holds for all $i \in [n]$ and $n \to \infty$, it is clear to see that $m = \infty$, and the proof is complete. □

Remark 35. It is easily seen that if $\operatorname{supp} G = [0, b]$, where $b < \infty$, then for $i > b$ we have $Z_n^i \xrightarrow{d} Z^i$, where Z_i is such that $\Pr(Z^i = b) = 1$. Therefore, $\lim_{n\to\infty} \mathbb{E}(M_n) = b$. Fig. 3a, where the expected value of M_n for a sample of random variables from the uniform distribution on $[0, 5]$ as a function of n is depicted, is an illustration of such a case. Please note the logarithmic scale on the X axis. As we may have expected, the variance of M_n, presented in Fig. 3b, tends to zero with increasing n.

Let us now consider some numerical results. Fig. 4a depicts the expected values of M_n as a function of sample size n for various exponential distributions $\mathcal{E}(\lambda)$. Please note the logarithmic scale on the X axis. Different values of the λ parameter, from 0.5 to 5, were considered. Note that in PAP the distribution parameters reflect a producer's ability to produce artifacts of high quality. The sample size describes the number of items created by a producer. Note that, as it was stated in Theorem 34, the expected values $\mathbb{E}(M_n)$ increase as n increases. As it was mentioned before, OM3 operators in PAP context are said to capture two dimensions of producers' quality – quality of their products and their productivity. However, producers that release artifacts with generally low rating may obtain the same evaluation as producers with high quality products simply by increasing the productivity. This may be easily noticed while investigating the expected value of M_n. For example, let us consider the expected evaluation of producers described by $\mathcal{E}(0.5)$, $\mathcal{E}(1)$ and $\mathcal{E}(1.5)$. Here, the expected quality of each of their products is described in terms of the expected values equal to 2, 1, and 2/3 and medians equal to 1.4, 0.7, and 0.4, respectively. However, the analyzed producers may get the same rating, for example 5, when they release about 62, 752 and over 9000 products, respectively. Similar

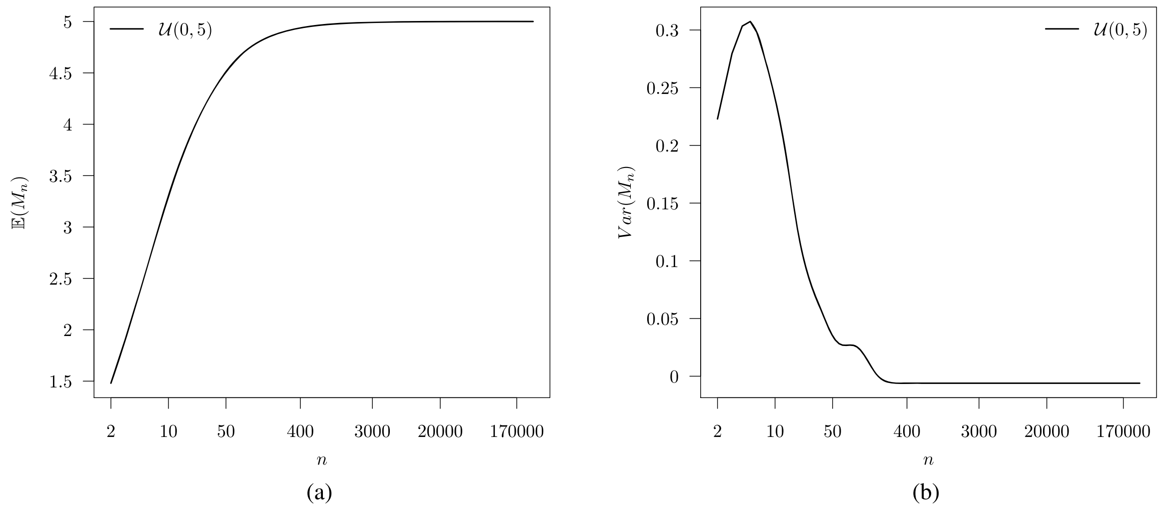

Fig. 3. Expected value (a) and variance (b) of OM3 operators for a sample of random variables from the uniform distribution on [0, 5], as a function of sample length, n.

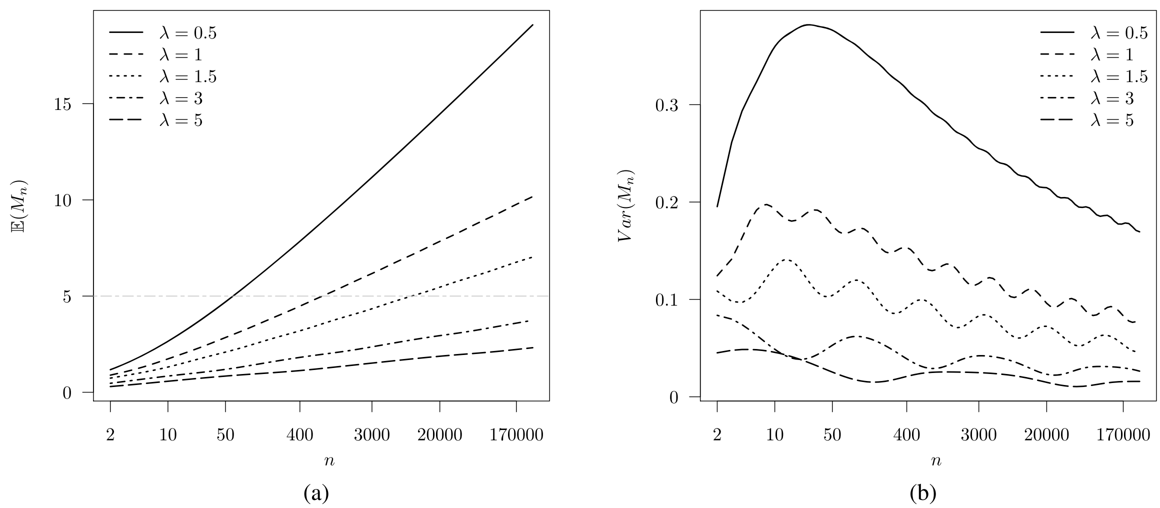

Fig. 4. Expected value (a) and variance (b) of OM3 operators for samples of random variables from various exponential distributions as a function of sample length, n.

conclusions can be made for a Paretian model, see Fig. 5a. Even though Pareto type II ($\mathcal{P}2(k,s)$) distribution is defined by two parameters – of shape k and scale s – it can be brought to a one parameter case by a proper input data transformation. Thus, in our investigation for the sake of simplicity, we assumed that $s = 1$. Please note that the expected value of $\mathcal{P}2(k,s)$ is undefined for $k \leqslant 1$.

Unlike the expected value of the variable M_n, its variance behaves in a more complicated way. Fig. 4b and Fig. 5b depict $\mathrm{Var}(M_n)$ as a function of sample size n for an exponential and Paretian model, respectively. Interestingly, for the exponential distribution, variance of M_n decreases as n increases. What is more, it is clear that this is not a monotonic function, but it seems to approach 0. Interestingly, in case of a Paretian model, on the other hand, opposite relation is observed, even for small values of $k \leqslant 2$ for which variance in $\mathcal{P}2$ does not exists.

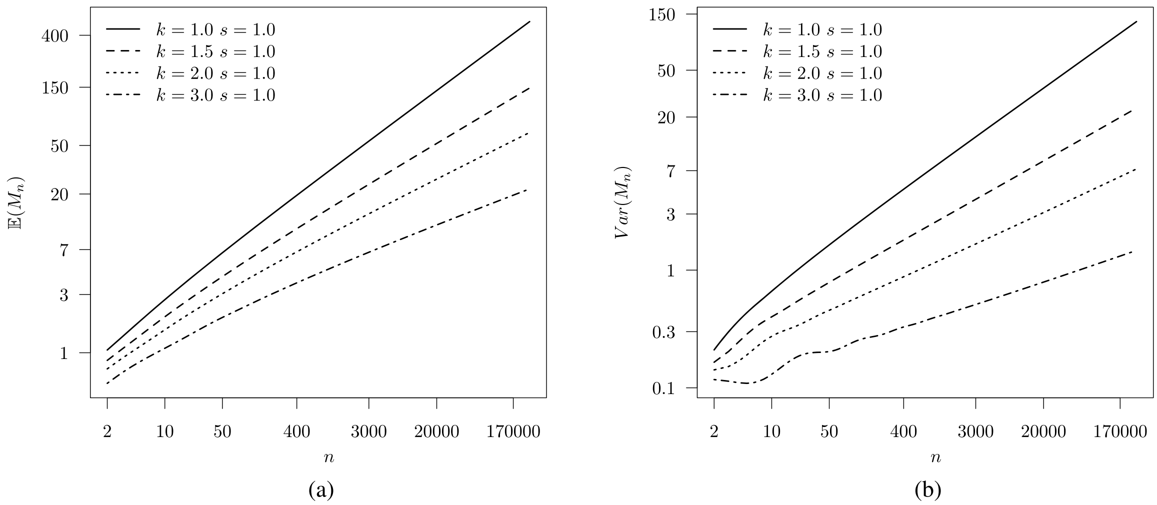

Fig. 5. Expected value (a) and variance (b) of OM3 operators for samples of random variables from various Pareto-type II distributions as a function of sample length, n.

5. Conclusions

Derivation of reliable and fair evaluation or ranking methods for information resources producers is an important challenge for information sciences and mathematics. We have recalled already known but also indicated some new interesting instances of the Producers Assessment Problem especially connected with dealing with information overload issue, e.g. in web and computer software quality assessment.

One of the possible ways to create tools for PAP is by utilizing the notion of aggregation operators. In fact, most of the proposals made in the informetric (especially bibliometric) literature fall into this class, even if their authors are not conscious about it. On the one hand, this is particularly interesting as it shows that the theory of aggregation has simple and intuitive foundations. On the other, special attention should be paid on informetricians' results as some of them may try to "reinvent the wheel".

In this paper we have focused especially on zero-insensitive OM3 aggregation operators, which naturally generalize Hirsch-like indices of impact. Many desirable properties were postulated or recalled. Quite unexpectedly, the OM3 operator that fulfills most of the properties is the Max function. It is known, however, that such a function does not take the productivity of a producer into account at all. None of the Hirsch-like indices (cf. Lemma 8) give a consistent ranking of tuples of producers, see Section 3.3.

All the OM3 operators suffer for the following drawback. For any x let i be such that $\mathsf{w}_n(x) \in (c_{i-1}, c_i]$. Then we have $\mathsf{M}_{\triangle,\mathbf{w}}(i * x, (n-i) * 0) = \mathsf{M}_{\triangle,\mathbf{w}}((i-1) * b, (n-i+1) * x)$. It is easily seen that the first aggregated vector is the minimal one giving this output value, and the latter one is the maximal (for fixed n). We may of course ask a question if the OM3 operators do not ignore too much information in an input vector. Our future work should be committed towards utilizing functions that result in a value $\in \mathbb{I}^k$ for some $k > 1$.

A similar axiomatic analysis, concerning different classes of aggregation operators, should be performed in the future. Researchers in some of the bibliometric papers suggested the utilization of "scoring rules", see [37,48], which may overcome the ranking inconsistency issues. These functions are nothing else than a subclass of modular aggregation operators, cf. [38]. Note that arity-monotonic modular operators may not fulfill F-insensitivity or max-insensitivity so easily.

What is even more, some recent results presented in [19] indicate that aggregation operators may not at all provide a proper way to assess producers. Such mathematical tools are known to describe well some characteristics of data sets (e.g. generalized means may be used to measure the so-called *central tendency*) or construct generalized connectives for fuzzy logic. However, we still need some stronger, formal arguments for or against their usage in PAP. Are e.g. the Hirsch-like indices only tools for stating that i items have quality of $\mathsf{w}(i)$ and nothing more?

Some interesting remarks may be also induced by analyzing the behavior of OM3 operators in a probabilistic setting.

Please note that the derived form of cumulative distribution function of zero-insensitive OM3 operators – which is defined in terms of regularized incomplete beta function – may often be only computed numerically. Firstly, all parameters of the regularized incomplete beta function depend on the x argument. Moreover, the continuity of a random variable's distribution does not imply the continuity of M_n, since it was shown that this function is usually not continuous at points from $[n]$. Another interesting relation was shown for asymptotic behavior of expected value of M_n. It turns out that for all random vectors following a distribution G, such that $\operatorname{supp} G = [0, \infty)$, expected value of M_n tends to infinity. Note that since many commonly used informetric tools, like the h-index, may be expressed as OM3 operators, this result has some interesting practical implications. No matter what is the expected quality measure of each input, increasing the productivity allows us to obtain arbitrarily high output valuation. In other words, when it comes to OM3 operators, productivity is of great importance.

On the other hand, numerical results concerning the variance of M_n show differ behavior depending on input data distribution.

Please note that the empirical results obtained in [10] indicate interesting directions worth more detailed investigation – like dependencies between different subclasses of OM3 operators, which can be modeled by copulas.

In [20], on the other hand, parametric statistical hypothesis test for the equality of probability distributions' parameters based on the difference between Hirsch's h-indices of two equal-length i.i.d. random samples were constructed. Please note that since h-index is a particular example of OM3 operators, thus the need of the generalization of this results seems to be quite natural.

Another interesting problem worth detailed investigation concerns modular operators. Even though many has been said in this matter, see e.g. [11], some new, interesting ideas may arrive when studying their probabilistic properties in an arity-dependent context, which is non-standard not only for the theory of aggregation, but also for probability theory.

Acknowledgements

Anna Cena's study was financed by research fellowship within Project "Information technologies: Research and their interdisciplinary applications", agreement UDA-POKL.04.01.01-00-051/10-00.

Marek Gagolewski's research was partially supported by the FNP START 2013 Scholarship from the Foundation For Polish Science.

References

[1] M. Abramovitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards Applied Mathematics Series, 1972.
[2] S. Alonso, F.J. Cabrerizo, E. Herrera-Viedma, F. Herrera, h-Index: a review focused on its variants, computation and standardization for different scientific fields, J. Informetr. 3 (2009) 273–289.
[3] K. Barcza, A. Telcs, Paretian publication patterns imply Paretian Hirsch index, Scientometrics 81 (2009) 513–519.
[4] G. Beliakov, S. James, Citation-based journal ranks: the use of fuzzy measures, Fuzzy Sets Syst. 167 (2011) 101–119.
[5] G. Beliakov, S. James, Stability of weighted penalty-based aggregation functions, Fuzzy Sets Syst. 226 (2013) 1–18.
[6] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Springer-Verlag, 2007.
[7] D. Bouyssou, T. Marchant, Ranking scientists and departments in a consistent manner, J. Am. Soc. Inf. Sci. Technol. 62 (2011) 1761–1769.
[8] M. Bras-Amorós, J. Domingo-Ferrer, V. Torra, A bibliometric index based on the collaboration distance between cited and citing authors, J. Informetr. 5 (2011) 248–264.
[9] A. Cena, M. Gagolewski, OM3: ordered maxitive, minitive, and modular aggregation operators – part I: axiomatic analysis under arity-dependence, in: H. Bustince, et al. (Eds.), Aggregation Functions in Theory and in Practise (AISC 228), Springer-Verlag, Heidelberg, 2013, pp. 93–103.
[10] A. Cena, M. Gagolewski, OM3: ordered maxitive, minitive, and modular aggregation operators – part II: a simulation study, in: H. Bustince, et al. (Eds.), Aggregation Functions in Theory and in Practise (AISC 228), Springer-Verlag, Heidelberg, 2013, pp. 105–115.
[11] H.A. David, H.N. Nagaraja, Order Statistics, Wiley, 2003.
[12] V.G. Deineko, G.J. Woeginger, A new family of scientific impact measures: the generalized Kosmulski-indices, Scientometrics 80 (2009) 819–826.
[13] D. Dubois, H. Prade, Semantics of quotient operators in fuzzy relational databases, Fuzzy Sets Syst. 78 (1996) 89–93.
[14] D. Dubois, H. Prade, C. Testemale, Weighted fuzzy pattern matching, Fuzzy Sets Syst. 28 (1988) 313–331.
[15] L. Egghe, Theory and practise of the g-index, Scientometrics 69 (2006) 131–152.
[16] F. Franceschini, D.A. Maisano, The Hirsch index in manufacturing and quality engineering, Qual. Reliab. Eng. Int. 25 (2009) 987–995.

[17] F. Franceschini, D.A. Maisano, Structured evaluation of the scientific output of academic research groups by recent h-based indicators, J. Informetr. 5 (2011) 64–74.
[18] M. Gagolewski, On the relationship between symmetric maxitive, minitive, and modular aggregation operators, Inf. Sci. 221 (2013) 170–180.
[19] M. Gagolewski, Scientific impact assessment cannot be fair, J. Informetr. 7 (2013) 792–802.
[20] M. Gagolewski, Statistical hypothesis test for the difference between Hirsch indices of two Pareto-distributed random samples, in: R. Kruse, et al. (Eds.), Synergies of Soft Computing and Statistics for Intelligent Data Analysis, in: Advances in Intelligent Systems and Computing, vol. 190, Springer-Verlag, 2013, pp. 359–367.
[21] M. Gagolewski, P. Grzegorzewski, Arity-monotonic extended aggregation operators, in: E. Hüllermeier, et al. (Eds.), Information Processing and Management of Uncertainty in Knowledge-Based Systems, in: Communications in Computer and Information Science, vol. 80, Springer-Verlag, 2010, pp. 693–702.
[22] M. Gagolewski, P. Grzegorzewski, S-statistics and their basic properties, in: C. Borgelt, et al. (Eds.), Combining Soft Computing and Statistical Methods in Data Analysis, Springer-Verlag, 2010, pp. 281–288.
[23] M. Gagolewski, P. Grzegorzewski, Possibilistic analysis of arity-monotonic aggregation operators and its relation to bibliometric impact assessment of individuals, Int. J. Approx. Reason. 52 (2011) 1312–1324.
[24] M. Gagolewski, R. Mesiar, Monotone measures and universal integrals in a uniform framework for the scientific impact assessment problem, Inf. Sci. 263 (2014) 166–174.
[25] R. Ghiselli Ricci, Asymptotically idempotent aggregation operators, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 17 (2009) 611–631.
[26] R. Ghiselli Ricci, R. Mesiar, Multi-attribute aggregation operators, Fuzzy Sets Syst. 181 (2011) 1–13.
[27] W. Glänzel, H-index concatenation, Scientometrics 77 (2008) 369–372.
[28] W. Glänzel, On some new bibliometric applications of statistics related to the h-index, Scientometrics 77 (2008) 187–196.
[29] M. Grabisch, J.L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Cambridge University Press, 2009.
[30] M. Grabisch, J.L. Marichal, R. Mesiar, E. Pap, Aggregation functions: construction methods, conjunctive, disjunctive and mixed classes, Inf. Sci. 181 (2011) 23–43.
[31] M. Grabisch, J.L. Marichal, R. Mesiar, E. Pap, Aggregation functions: means, Inf. Sci. 181 (2011) 1–22.
[32] J.E. Hirsch, An index to quantify individual's scientific research output, Proc. Natl. Acad. Sci. USA 102 (2005) 16569–16572.
[33] R. Hovden, Bibliometrics for Internet media: applying the h-index to YouTube, J. Am. Soc. Inf. Sci. Technol. 64 (2013) 2326–2331.
[34] Y.A. Hwang, An axiomatization of the Hirsch-index without adopting monotonicity, Appl. Math. Inf. Sci. 7 (2013) 1317–1322.
[35] E. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral, IEEE Trans. Fuzzy Syst. 18 (2010) 178–187.
[36] M. Kosmulski, A new Hirsch-type index saves time and works equally well as the original h-index, ISSI Newsl. 2 (2006) 4–6.
[37] T. Marchant, Score-based bibliometric rankings of authors, J. Am. Soc. Inf. Sci. Technol. 60 (2009) 1132–1137.
[38] R. Mesiar, A. Mesiarová-Zemánková, The ordered modular averages, IEEE Trans. Fuzzy Syst. 19 (2011) 42–50.
[39] A. Miroiu, Axiomatizing the Hirsch index: quantity and quality disjoined, J. Informetr. 7 (2013) 10–15.
[40] A. Quesada, Monotonicity and the Hirsch index, J. Informetr. 3 (2009) 158–160.
[41] A. Quesada, More axiomatics for the Hirsch index, Scientometrics 82 (2010) 413–418.
[42] A. Quesada, Axiomatics for the Hirsch index and the Egghe index, J. Informetr. 5 (2011) 476–480.
[43] A. Quesada, Further characterizations of the Hirsch index, Scientometrics 87 (2011) 107–114.
[44] R. Rousseau, Woeginger's axiomatisation of the h-index and its relation to the g-index, the $h(2)$-index and the r^2-index, J. Informetr. 2 (2008) 335–340.
[45] R.J. Serfling, Approximation Theorems of Mathematical Statistics, John Wiley & Sons, New York, 1980.
[46] M. Sugeno, Theory of fuzzy integrals and its applications, Ph.D. thesis, Tokyo Institute of Technology, 1974.
[47] V. Torra, Y. Narukawa, The h-index and the number of citations: two fuzzy integrals, IEEE Trans. Fuzzy Syst. 16 (2008) 795–797.
[48] L. Waltman, N.J. van Eck, The inconsistency of the h-index, J. Am. Soc. Inf. Sci. Technol. 63 (2012) 406–415.
[49] G.J. Woeginger, An axiomatic analysis of Egghe's g-index, J. Informetr. 2 (2008) 364–368.
[50] G.J. Woeginger, An axiomatic characterization of the Hirsch-index, Math. Soc. Sci. 56 (2008) 224–232.
[51] G.J. Woeginger, A symmetry axiom for scientific impact indices, J. Informetr. 2 (2008) 298–303.