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MULTIPLE CITATION PATTERNS IN SCIENTIFIC LITERATURE: THE CIRCLE AND HILL MODELS

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Summary—The concept of tri-citation is introduced, as a logical extension of co-citation, and a geometrical model (the circle model) is devised to account for these and all other forms of multiple citation. The model is used to predict distances between documents which can be scaled metrically in two dimensional space. The model is also used to predict observed tri-citation frequencies among a set of six documents in particle physics. A further refinement of the model is suggested (the hill model) which promises to be more flexible. Implications of such spatial models for the representation of clusters of highly cited documents are discussed.

CO-CITATION frequency is defined as the number of times two documents have been cited together in later documents. This simple matching coefficient has been applied recently in a number of bibliometric studies[1, 2, 3, 4]. It has proved particularly successful in clustering experiments aimed at automatically generating compact sets of documents for narrowly defined specialty literatures. These experiments have led to attempts to "map science" as a network of specialties and to create spatial models of specialty literatures using multidimensional scaling techniques[5].

This paper introduces the notion of "tri-citation" as a logical extension of co-citation, and explores a geometrical model for all forms of multiple citation. The model appears to have important implications for the spatial modeling of scientific specialties. It assumes that there exists, in some sense, a subject "space" which is occupied by documents and in which the distance between documents in the space is interpretable as subject similarity, i.e. the closer the documents the more similar their content. The model requires that frequently cited documents be regarded as extended objects rather than as geometrical points, and a cluster of such documents as a system of overlapping regions.

CO-CITATION AND TRI-CITATION

Just as co-citation may be defined as the intersection of two sets of citing papers, tri-citation is the intersection of three such sets. If A is the set of papers citing document a, and B the set citing b, the frequency of co-citation of a and b is $n(A \cap B)$. If a third set of papers C, citing document c, is introduced, then the frequency of tri-citation of a, b and c is $n(A \cap B \cap C)$. In practice, tri-citation frequency can be determined by counting the number of identical citing papers listed under three different cited document entries in the *Science Citation Index* (*SCI*). Clearly, it is a simple step to generalize co- and tri-citation to the intersection of an arbitrary number of sets of citing papers. Given N cited papers, let A_i, $i = 1, 2, \ldots N$, denote the sets of citing papers. Then $n(\cap A_i)$, $i = 1, 2, \ldots N$, is the number of papers citing all N papers. When N is large the members of $(\cap A_i)$ will be predominantly review papers. A cluster of N intersecting cited papers will have a total of $n(\cup A_i)$, $i = 1, 2, \ldots N$ citing papers associated with it. Such clusters of highly cited documents would display a full range of multiple citation patterns. In the following sections, the models developed could equally apply to all forms of multiple citation, but examples will be restricted to co-citation and tri-citation, which are the simplest forms.

REPRESENTATION OF A CITED DOCUMENT AS A CIRCULAR AREA: THE CIRCLE MODEL

The central idea of this paper is that a cited document may be represented as a circle whose area is proportional to the number of papers which cite it, i.e. citation frequency = πr^2. This means that each citing paper occupies a unit area of two-dimensional space, and that the distribution of citing papers is uniform in the plane. For the moment, no attempt will be made to rationalize the choice of the circle as a geometric form, considering it as only one of many possible modes of representation, the validity of which can only be determined empirically.

The present model leads to the following interpretation of co-citation and tri-citation frequency: If two documents are co-cited, the frequency of co-citation will be equal to the area of intersection of the two circular areas; if three documents are tri-cited, the frequency of tri-citation will equal the area of tri-section of the three circles (see Figs. 3 and 4). The distance between two cited documents may be interpreted as the distance between the centers of the two overlapping circles. Since for a group of three documents these distances are completely determined by the citation frequencies and the co-citation frequencies, the area of tri-section is determined, and this area may be used as a predicted value for the tri-citation count from the *SCI.*

In the following, two tests of the circle model will be discussed: (1) by determining the self-consistency of distances among a set of documents, all of which are co-cited; and (2) by comparing areas of tri-section of three circles with observed tri-citation frequencies from the *SCI.*

Cluster of six documents in particle physics

As a test case a set of six documents in a specialty of particle physics was selected (see Table 1). These documents contain some of the most significant and influential work on strong interaction physics which has appeared in recent years. A 1962 Amati paper contained the earliest discussion of a concept known as scaling. In 1969 two of the most influential physicists of our time, R. P. Feynman, and C. N. Yang (third author on the Benecke paper) proposed hypotheses to explain scaling. Feynman's version made use of the idea of a particle within the proton called the parton. In 1970 a paper by Mueller appeared which worked out the technical and mathematical implications of these hypotheses in a way which opened up the area for a large number of physicists. Papers which followed depended heavily on these founding papers, and represented a working out of the implications of the basic theory. Some of these papers then became highly co-cited with the founding papers (e.g. the DeTar and Caneschi papers).

The full bibliographic reference for each of the documents is given in Table 1 along with its citation frequency from the 1972 *SCI* and its radius based on the circle model. All of the papers were frequently cited in 1972 and all were co-cited with each other. Hence it was possible to calculate all 15 distances among the six documents. The calculations were performed by successive approximations using the formulas derived in Appendix I, which express the area of intersection of two circles (co-citation frequency) as a function of the areas of the two circles (citation frequencies) and the distance between their centers. The calculated distances are given below the diagonal in the matrix in Table 2 (the upper entry in each cell), and the co-citation frequencies among the six documents are given above the diagonal.

The lower-half-matrix of distances is used as input to a multidimensional scaling program[6], as described in an earlier paper[7]. The program begins by specifying an initial (random) configuration for the six papers. The distances among these papers are calculated, and compared with what the distances should be. A badness-of-fit measure called "stress" is calculated (actually, "stress II" is used, which is a slight modification of Kruskal's original formula[8]), and the points are moved to new locations. New distances among them are calculated and a new

Table 1. Six papers in particle physics

Document	Code	Citation Frequency	Radius
D. Amati, A. Stanghellini and S. Fubini: Theory of high-energy scattering and multiple production. *Nuovo Cim.* 1962, **26** (5), 896–954.	A	88	5·29
J. Benecke, T. T. Chou, C. N. Yang and E. Yen: Hypothesis of limiting fragmentation in high-energy collisions. *Phys. Rev.* 1969, **188** (5), 2159–2169.	B	127	6·36
L. Caneschi and A. Pignotti: Multi-regge baryon exchange and central interactions: *Phys. Rev. Lett.* 1969, **22** (22), 1219–1223.	C	39	3·52
C. E. DeTar, C. E. Jones, F. E. Low, J. H. Weis, J. E. Young and C. I. Tan: Helicity poles, triple regge behavior and single particle spectra in high energy collisions: *Phys. Rev. Lett.* 1971, **26** (11), 675–676.	D	76	4·92
R. P. Feynman: Very high-energy collisions of hadrons: *Phys. Rev. Lett.* 1969, **23** (24), 1415–1417.	F	187	7·72
A. H. Mueller: 0(21) analysis of single-particle spectra at high-energy: *Phys. Rev. D.* 1970, **2** (12), 2963–2968.	M	142	6·72

Table 2. Circle model distances: calculated and MDSCAL

	A	B	C	D	F	M
A		20	9	14	27	24
B	8.15 9.13		26	20	93	53
C	6.51 5.38	5.09 4.43		18	27	22
D	7.34 6.73	7.72 7.86	4.69 4.11		25	45
F	8.81 8.90	4.40 3.21	6.38 5.88	8.58 9.94		52
M	8.08 9.90	6.50 5.97	6.05 5.02	5.35 4.34	8.19 8.95	

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stress is calculated. This process is repeated until a minimum value for stress is obtained.

A metric scaling in two dimensions was carried out for the six documents attempting to fit the distances in Table 2 exactly (upper entry in each cell). The final fitted distances among the documents is also given in Table 2 (lower entry in each cell). The final value for stress (stress II) was 0·425. The scaled distances are plotted in Fig. 1 as distances between centers of circular areas which represent the documents according to the letter code in Table 1.

The structure is interesting in view of the earlier discussion of the history of this area. The Amati paper which contains the original discussion of scaling is relatively far removed from the others, as if relating equally to all succeeding developments. The Feynman and Yang (Benecke) hypotheses are the closest together of all and are located on one extreme side of Amati. The technically important paper by Mueller is somewhat separated from the Feynman–Yang group to Amati's extreme right. The working-out papers of DeTar and Caneschi have their strongest orientation toward Mueller. Caneschi is at the center, but his position is central only in that his work relates about equally to all the others. In general, in this representation the founding papers are located at the periphery of the map and form a framework within which all other work then falls.

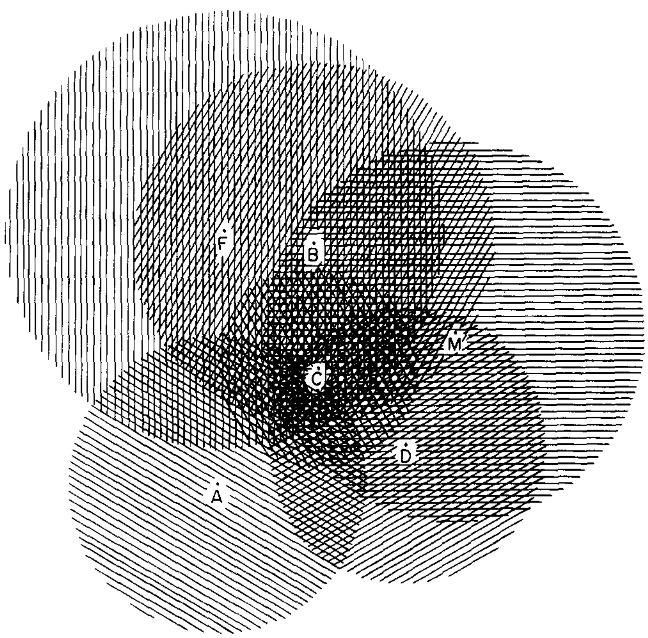

Fig. 1. Two-space configuration of six papers in particle physics: circle model.

The overlapping cross-hatching graphically illustrates the regions of multiple citation, i.e. co-citation, tri-citation, and so on. The small region where all six circles overlap is occupied by papers which have cited all six. These papers are of the review type. The size of the overlap areas could, in principle, be measured from the figure and compared with the counts obtained from the *SCI*. This approach was not taken but instead the tri-citation areas were calculated exactly using trigonometry, taking each set of three papers separately.

Prediction of tri-citation frequencies using the circle model

As mentioned previously, once the distances between the centers of the document circles have been established, the area of tri-section, and hence the frequency of tri-citation, is determined. These predicted values can then be compared with those observed by counting matching citing papers among three cited document entries in the *SCI*. There are a total of 20 combinations of six documents taken three at a time. For each group of three documents a value for the area of trisection was calculated according to the formulas given in Appendix II. This area is given as a function only of the distances among the three document circles and the radii of the circles. Using the 1972 *SCI* the entries for the three documents were matched and the number of citing papers common to all three was counted. The results are presented in Table 3. Each of the 20 sets of three documents is represented as a combination of three letters according to the codes in Table 1. The calculated area of tri-section is given in the "Predicted" column, and under "Observed", the tri-citation count from the 1972 *SCI*. The final column gives the chi-square statistic for comparison of predicted and observed values. For 20 d.f., the agreement is at the 99 per cent confidence level.

Table 3. Circle model: predicted and observed tri-citation counts

	Tri-citation	Observed	Predicted	Chi square
1	BFM	41	39·81	0·036
2	BCF	24	22·60	0·087
3	BDF	19	16·25	0·47
4	ABF	18	19·57	0·13
5	DFM	18	18·62	0·021
6	CFM	17	16·31	0·029
7	BCM	16	16·94	0·052
8	BDM	15	17·07	0·25
9	CDF	14	11·97	0·34
10	AFM	13	12·13	0·062
11	BCD	13	11·27	0·27
12	CDM	11	13·65	0·51
13	ABM	9	11·37	0·49
14	ADM	8	9·72	0·30
15	ACF	7	6·96	0·00023
16	ABC	7	6·60	0·024
17	ACM	6	5·86	0·0033
18	ADF	5	6·31	0·27
19	ABD	3	4·94	0·76
20	ACD	2	4·56	1·44
				5·54

THE HILL MODEL

The success of the circle model in predicting tri-citation counts strongly supports the interpretation of a cited document as an extended area in subject space, rather than as a geometrical point. The circle model, however, has the undesirable feature of cutting off abruptly; all citing papers must be no more than a radius away from the center of the circle.

Another model which allows citing papers to be relatively far removed from the center is the hill model. The hill is a much more general solution to the problem of representing a cited document in subject space [9]. In this model the distribution of citing papers is described by a two dimensional normal probability density function of the form:

$$\frac{c_i}{2\pi\sigma^2}\cdot \exp -\frac{x^2+y^2}{2\sigma^2}$$

where σ is the standard deviation which for the time being is assumed the same for all documents, and c_i is the citation frequency of document i. This function describes the density of citing papers about the central point (in this case the origin of the x–y coordinate system). In contrast to the uniform density of the circle model, citing papers are concentrated at the center and thin out as the distance from the center increases. Since the total volume under this probability surface is c_i, the citation frequency, any portion of the volume defined by a region on the x–y plane is equal to the number of citing papers within that region.

The co-citation frequency is defined as the volume beneath the surfaces of two intersecting hills which are separated by a distance, d. Hence, the co-citation, c_{ij}, between documents i and j can be expressed as the sum of two volume integrals:

$$c_{ij} = \frac{c_i}{2\pi\sigma^2}\int_{x_o}^{+\infty}\int_{-\infty}^{+\infty}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)dy\,dx + \frac{c_j}{2\pi\sigma^2}\int_{x_o}^{-\infty}\int_{-\infty}^{+\infty}\exp\left(-\frac{(x-d)^2+y^2}{2\sigma^2}\right)dy\,dx.$$

The lower limit of integration, x_o, is obtained by setting the two hill functions equal to one another.

$$x_o = \frac{d}{2} + \frac{\sigma^2(\ln c_i - \ln c_j)}{d}.$$

After integrating with respect to y, the equation can be reduced to the simple form:

$$c_{ij} = c_i\left(1 - F\left(\frac{x_o}{\sigma}\right)\right) + c_j\left(1 - F\left(\frac{d - x_o}{\sigma}\right)\right)$$

where

$$F(z) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{z}\exp\left(-\frac{t^2}{2}\right)dt.$$

$F(z)$ can be found from tables of the normal probability function. Hence, a value for the distance, d, can be found given the three numbers c_{ij}, c_i, c_j.

To test the hill model the co-citation data on the six papers in particle physics were used. First, the distances between the centers of the document hills were calculated assuming a standard deviation of one for all hills. The resulting 15 distances are given in Table 4 (upper entry in

Table 4. Hill model distances: calculated and MDSCAL

	A	B	C	D	F
B	2.6 2.7				
C	2.8 3.4	1.6 1.3			
D	2.7 3.0	2.5 2.6	1.9 1.9		
F	2.5 2.4	1.0 .5	1.8 1.8	2.5 2.9	
M	2.5 2.1	1.7 1.7	1.9 1.5	1.5 1.0	2.0 1.9

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each cell) to two significant figures. By reference to Table 2 it is evident that for the same citation and co-citation frequencies, the range of distances is greater for the hill model than for the circle model. The ratio of the longest to the shortest hill distances is 2·8, while the same ratio for the circle model is 2·0. This will in general be the case because the closest two circles can get together is $(r_i - r_j)$, when the smaller circle is just within the bigger one; while two hills can approach zero distance for the case of complete overlap. At the other extreme, the limiting distance two circles can be apart is $(r_i + r_j)$, when they are just touching; while two hills may continue to move apart as long as the volume of overlap does not fall below one.

The calculated distances were then used as input to the multidimensional scaling program, as described previously. The fitted distances are given in Table 4 (lower entry in each cell). The stress (formula II) was 0·403, a slight improvement over the previously obtained stress value for the circle model. Because the citing papers are not evenly distributed over the plane, the six papers are properly represented as a contour map as in Fig. 2. Along a given contour line the density of citing papers is constant. At the peak of the Feynman hill, the density is 27 citing papers per unit area. At the periphery of the map the density is three citing papers per unit area. In general, the cited papers have roughly the same relative locations as in the circle model, with the one major exception of the Caneschi paper which has moved out of the center of the configuration to the periphery.

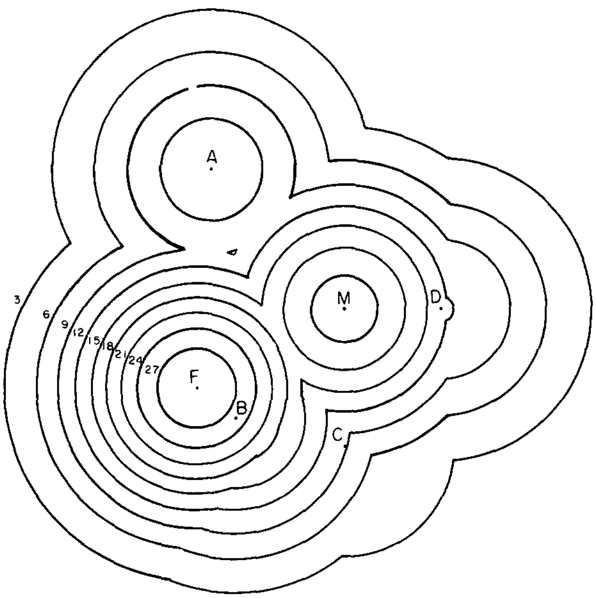

Fig. 2. Contour map of six papers in particle physics: hill model.

DISCUSSION

The problem posed in this paper is how to represent a frequently cited document in subject space, and, by extension, a cluster of such documents. The solution proposed is to allow the papers citing a particular document to expand from their most primitive representation as a geometric point, until they are distributed in some fashion about a central location which may be taken as the position of the cited document. A distribution with circular symmetry makes sense since there is no reason to assume preferred directions in subject space.

The two models which seem most promising are the circle model, with a uniform density of citing papers, and the hill model requiring a higher concentration toward the center. Both models tend to give longer distances when the papers are highly cited and shorter distances when they are less cited for the case of constant co-citation. This is because the models treat cited papers as group attributes and the group occupies space according to its number. This was evident in the particle physics cluster where the highly cited (and founding) papers were in effect the poles of the map and defined its outer extremities.

The circle model has shown a remarkable ability to predict tri-citation frequencies, and an adequate but far from perfect fit for metric scaling in two dimensions. The hill model, when the standard deviations of all documents were considered equal, gave a slightly improved stress value for scaling in two dimensions. A test of the hill model using the tricitation counts has not yet been attempted. Obviously, the next step would be to allow the standard deviation to be different for each document. As Kruskal has suggested[9], σ could measure a document's breadth or narrowness of impact: methodological papers might have a large σ and highly specialized papers a small σ. A rational means for assigning σ values would be the total number of papers a given paper is linked with through co-citation, which is an indication of the breadth of impact of the paper. The hill model representation of a cluster of documents would then resemble a landscape of the most varied sort, having both slender peaks and rolling hills. If successive annual cumulations of citation data were used to study the cluster, a shifting landscape would be observed, with the hills and peaks changing in size and shape and shifting relative to one another. Studies of this kind, it is hoped, will reveal something of the nature of change in scientific specialties.

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APPENDIX I: DISTANCE BETWEEN TWO INTERSECTING CIRCLES

Since the frequency of co-citation of two papers has been interpreted as the area of overlap of two circles whose areas are equal to the citation frequencies of the papers, a formula is required which relates this area of intersection to the distance between the centers of the circles. This distance will then be taken as the distance between the two papers in subject space.

Given two intersecting circles (see Fig. 3) whose centers are at points A and B, an expression is sought relating the area of

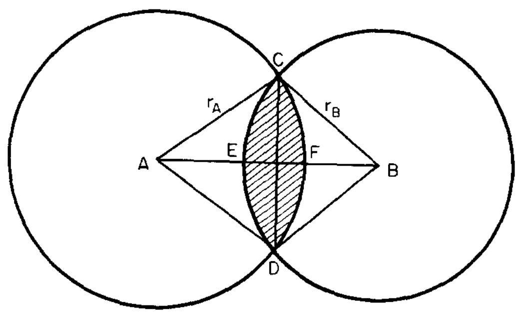

Fig. 3. Co-citation area.

intersection (shaded in the figure) to the distance between the centers, d_{AB}, and the radii, r_A and r_B, of the two circles:

$$\begin{aligned}\text{Area intersection} &= \text{area segment CED} + \text{area segment CFD}\\ &= \text{area sector ACFD} + \text{area sector BCED}\\ &\quad - \text{area quadrilateral ACBD}\\ &= \frac{\angle CAD}{2}\cdot r_A^2 + \frac{\angle CBD}{2}\cdot r_B^2 - d_{AB}\cdot r_A\cdot \sin\left(\frac{\angle CAD}{2}\right)\end{aligned}$$

where

$$\frac{\angle CAD}{2} = \cos^{-1}\left(\frac{r_A^2 - r_B^2 + d_{AB}^2}{2\cdot r_A\cdot d_{AB}}\right) \quad \text{(in radians)}$$

$$\frac{\angle CBD}{2} = \cos^{-1}\left(\frac{r_B^2 - r_A^2 + d_{AB}^2}{2\cdot r_B\cdot d_{AB}}\right)$$

$$\sin\left(\frac{\angle CAD}{2}\right) = \left[1 - \cos^2\left(\frac{\angle CAD}{2}\right)\right]^{1/2} = \left[1 - \left(\frac{r_A^2 - r_B^2 + d_{AB}^2}{2\cdot r_A\cdot d_{AB}}\right)^2\right]^{1/2}.$$

The desired expression is obtained by combining these four equations. The result, however, is not simple from a computational point of view, due to the impossibility of solving for the distance, d_{AB}, as a function of the other variables. Therefore, in the calculations of distance a computer program involving successive approximations to d_{AB} has been employed.

APPENDIX II: AREA OF TRI-SECTION OF THREE CIRCLES (TRI-CITATION)

The frequency of tri-citation of three papers has been interpreted as the area of tri-section of three circles. Given three circles with centers at A, B, and C all of which overlap (see Fig. 4), the distances between the centers of the circles d_{AB}, d_{BC} and d_{CA} can be determined from co-citation frequencies. An expression for the area of tri-section is sought as a function of the three distances and the radii of the three circles r_A, r_B, and r_C. These areas can then be compared to the tri-citation counts from the *SCI*.

$$\begin{aligned}\text{Area tri-section} &= \text{area triangle DEF} + \text{area segment DE}\\ &\quad + \text{area segment EF} + \text{area segment FD}\\ \text{Area triangle DEF} &= [s(s - d_{DE})(s - d_{EF})(s - d_{FD})]^{1/2}\end{aligned}$$

where

$$s = \tfrac{1}{2}(d_{DE} + d_{EF} + d_{FD})$$

$$d_{DE} = 2\cdot r_C\cdot \sin\left(\frac{\angle DCE}{2}\right)$$

$$d_{EF} = 2\cdot r_A\cdot \sin\left(\frac{\angle EAF}{2}\right)$$

$$d_{FD} = 2\cdot r_B\cdot \sin\left(\frac{\angle DBF}{2}\right)$$

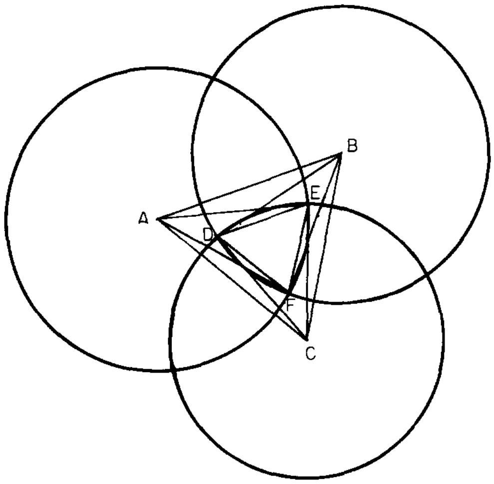

Fig. 4. Tri-citation area.

$$\text{Area segment DE} = \tfrac{1}{2} \cdot r_C^{\,2} (\angle \text{DCE} - \sin \angle \text{DCE})$$

$$\text{Area segment EF} = \tfrac{1}{2} \cdot r_A^{\,2} (\angle \text{EAF} - \sin \angle \text{EAF})$$

$$\text{Area segment FD} = \tfrac{1}{2} \cdot r_B^{\,2} (\angle \text{DBF} - \sin \angle \text{DBF}).$$

The angles can be expressed in terms of known distances and radii using the law of cosines:

$$\angle \text{DCE} = \angle \text{BCD} + \angle \text{ACE} - \angle \text{ACB}$$

$$= \cos^{-1}\left(\frac{r_C^{\,2} + d_{BC}^2 - r_B^{\,2}}{2 \cdot r_C \cdot d_{BC}}\right) + \cos^{-1}\left(\frac{r_C^{\,2} + d_{AC}^2 - r_A^{\,2}}{2 \cdot r_C \cdot d_{AC}}\right) - \cos^{-1}\left(\frac{d_{AC}^2 + d_{BC}^2 - d_{AB}^2}{2 \cdot d_{AC} \cdot d_{BC}}\right)$$

$$\angle \text{EAF} = \angle \text{BAF} + \angle \text{CAE} - \angle \text{BAC}$$

$$= \cos^{-1}\left(\frac{r_A^{\,2} + d_{AB}^2 - r_B^{\,2}}{2 \cdot r_A \cdot d_{AB}}\right) + \cos^{-1}\left(\frac{r_A^{\,2} + d_{AC}^2 - r_C^{\,2}}{2 \cdot r_A \cdot d_{AC}}\right) - \cos^{-1}\left(\frac{d_{AB}^2 + d_{AC}^2 - d_{BC}^2}{2 \cdot d_{AB} \cdot d_{AC}}\right)$$

$$\angle \text{DBF} = \angle \text{ABF} + \angle \text{CBD} - \angle \text{ABC}$$

$$= \cos^{-1}\left(\frac{r_B^{\,2} + d_{AB}^2 - r_A^{\,2}}{2 \cdot r_B \cdot d_{AB}}\right) + \cos^{-1}\left(\frac{r_B^{\,2} + d_{BC}^2 - r_C^{\,2}}{2 \cdot r_B \cdot d_{BC}}\right) - \cos^{-1}\left(\frac{d_{AB}^2 + d_{BC}^2 - d_{AC}^2}{2 \cdot d_{AB} \cdot d_{BC}}\right).$$