Mathematical properties of Q-measures

Ronald Rousseau [a,b,c], Yuxian Liu [d], Raf Guns [a,*]

[a] *Institute for Education and Information Sciences, IBW, University of Antwerp, Venusstraat 35, Antwerp B-2000, Belgium*
[b] *Faculty of Engineering Technology, VIVES-KHBO (Association KU Leuven), Zeedijk 101, Oostende B-8400, Belgium*
[c] *KU Leuven, Leuven B-3000, Belgium*
[d] *Library of Tongji University, Tongji University, Siping Street, 1239, Shanghai 200092, China*

ARTICLE INFO

ABSTRACT

Q-measures are network indicators that gauge a node's brokerage role between different groups in the network. Previous studies have focused on their definition for different network types and their practical application. Little attention has, however, been paid to their theoretical and mathematical characterization. In this article we contribute to a better understanding of Q-measures by studying some of their mathematical properties in the context of unweighted, undirected networks. An external Q-measure complementing the previously defined local and global Q-measure is introduced. We prove a number of relations between the values of the global, the local and the external Q-measure and betweenness centrality, and show how the global Q-measure can be rewritten as a convex decomposition of the local and external Q-measures. Furthermore, we formally characterize when Q-measures obtain their maximal value. It turns out that this is only possible in a limited number of very specific circumstances.

© 2013 Elsevier Ltd. All rights reserved.

1. Introduction

Many research topics are nowadays studied from a network perspective. This is not only the case in the life sciences and the physical sciences, but also in the social sciences and the humanities (Goh et al., 2007; Newman, 2003; Risse, 2000; Wasserman & Faust, 1994). Large-scale analyses of so-called complex networks reveal that the same structural features – such as skewed degree distributions and local clustering – can emerge in different fields. This underlines the importance of network studies.

The field of informetrics is no exception to this trend. Indeed, topics such as collaboration, diffusion and citation have been studied from the perspective of social network analysis (see e.g., Franceschet, 2012; Leydesdorff, 2007; Liu, Rafols, & Rousseau, 2012; Liu, Rousseau, & Guns, 2013; Otte & Rousseau, 2002; Rousseau, Liu, & Ye, 2012; Yan & Ding, 2009). In these studies special attention is often paid to ranking entities (authors, journals, papers, etc.) according to one or more network indicators. The simplest example is ranking by (in-)degree; in the case of citation networks this is equivalent to ranking by number of citations. Moreover, the field of social network analysis (SNA) has introduced many centrality indicators that gauge the importance of a node as an element in a network (Wasserman & Faust, 1994). A well-studied example is betweenness centrality (Anthonisse, 1971; Freeman, 1977), which measures the extent to which shortest paths between nodes in the network pass through a given node. As such, it characterizes this node's control over the information flow through the network.

* Corresponding author. Tel.: +32 32654510.
E-mail addresses: ronald.rousseau@ua.ac.be (R. Rousseau), yxliu@lib.tongji.edu.cn (Y. Liu), raf.guns@ua.ac.be (R. Guns).

Depending on the network, it may happen that each node belongs to a larger subgroup. For instance, authors belong to a department, a university or a country; articles belong to a journal; journals belong to a publisher's portfolio or a scientific discipline. Typically, some nodes are only important within their own group, whereas others are 'brokers' or 'bridges' between several groups in the network. Flom, Friedman, Strauss, and Neaigus (2004) introduced a new indicator, called *Q-measure*, for the brokerage role of nodes between two groups in a connected, undirected, unweighted network. This original definition was later extended to networks with more than two groups, as well as weighted and directed networks (Guns & Rousseau, 2009; Rousseau & Zhang, 2008). In the case of networks with three or more groups, a global as well as a local Q-measure have been introduced (see Section 2). Q-measures, where only shortest paths between nodes from different groups are taken into account, are another variant of betweenness centrality (Brandes, 2008; Flom et al., 2004).

In this article we aim to contribute to a better understanding of Q-measures by studying some of their mathematical properties in unweighted, undirected networks. The remainder of this article is organized as follows. The next section reviews the definitions of betweenness centrality and global and local Q-measures. Section 3 constitutes the main theoretical contribution of this article by introducing external Q-measures and studying the precise relations between betweenness centrality and Q-measures. In particular we present a convex decomposition of the global Q-measure into a local and an external Q-measure. In Section 4 we present a characterization of nodes with a maximum global, local or external Q-measure (i.e., equal to one if normalization is applied). Finally, the last section presents the conclusions.

2. Q-measures and betweenness centrality: definitions

We assume that we have a network $N = (V, E)$, consisting of a set V of nodes or vertices and a set E of links or edges. A shortest path or geodesic between nodes g and h is denoted as $\gamma_{g,h}$. A geodesic between g and h that passes through a ($a \neq g$, $a \neq h$) is denoted as $\gamma_{g,h}(a)$.

Betweenness centrality is a measure characterizing the importance of a given node in establishing short pathways between other nodes (Anthonisse, 1971; Freeman, 1977). Mathematically, betweenness centrality of a node a is expressed as

$$\sum_{g,h \in V} \frac{p_{g,h}(a)}{p_{g,h}} \tag{1}$$

where $p_{g,h} = |\gamma_{g,h}|$ is the number of geodesics between nodes g and h ($g \neq h$) and $p_{g,h}(a) = |\gamma_{g,h}(a)|$ is the number of geodesics between nodes g and h that pass through a. Normalizing formula (1) leads to a number between 0 and 1. For an undirected network with n nodes this normalized form leads to formula (2):

$$C_B(a) = \frac{2}{(n-1)(n-2)} \sum_{g,h \in V} \frac{p_{g,h}(a)}{p_{g,h}} \tag{2}$$

Betweenness centrality has become one of the standard centrality measures in social network analysis, along with degree centrality, closeness centrality and eigenvector centrality (Wasserman & Faust, 1994). Many variants of betweenness centrality have since been proposed, such as group betweenness centrality (Everett & Borgatti, 1999) and edge betweenness centrality (Girvan & Newman, 2002).

In this article we focus mainly on the so-called Q-measures, originally introduced by Flom et al. (2004). If the network consists of two groups G (consisting of m nodes) and H (consisting of n nodes), then the Q-measure of node a is defined as:

$$Q(a) = \frac{1}{TP} \sum_{\substack{g \in G \\ h \in H}} \frac{p_{g,h}(a)}{p_{g,h}} \tag{3}$$

Here, TP denotes the total number of possible pairs of nodes from the two groups, not including a. If $a \in G$, then $TP = (m-1) \cdot n$ and if $a \in H$, then $TP = (n-1) \cdot m$.

Q-measures have subsequently been studied and applied in Chen and Rousseau (2008), Guns and Liu (2010), Guns, Liu, and Mahbuba (2011), Guns and Rousseau (2009), Rousseau (2005), and Rousseau and Zhang (2008). Rousseau and Zhang (2008) introduced Q-measures for networks with directed and weighted links. Guns and Rousseau (2009) expanded the definition to networks with any finite number of groups and showed that in this case one can define both a global and a local variant. An application of the concept of Q-measures is provided by Guns et al. (2011). These authors study a collaboration network of 1129 researchers from different countries, in the fields of bibliometrics, informetrics, webmetrics, and scientometrics during the period 1990–2009.
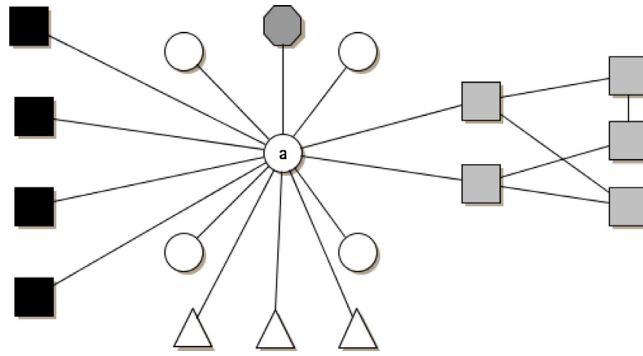
Fig. 1. Network with 5 groups. Node a has local and global Q-value equal to 1.

We now recall the definitions of global and local Q-measures. Assume that there are S groups $(2 \le S < +\infty)$ $G_1, \ldots, G_S$; the number of nodes in group G_i is denoted as m_i. The global Q-measure of node a is then defined as:

$$Q_G(a) = \frac{1}{C} \sum_{k,l} \left(\frac{1}{TP_{k,l}} \sum_{\substack{g \in G_k \\ h \in G_l}} \frac{p_{g,h}(a)}{p_{g,h}} \right) \tag{4}$$

where $C = \binom{S}{2} = (S(S-1))/2$ is the number of ways to choose two different groups, irrespective of their order.

In the context of Q-measures for several groups $1/(TP_{k,l}) \sum_{\substack{g \in G_k \\ h \in G_l}} (p_{g,h}(a))/p_{g,h}$ is denoted $Q_{k,l}(a) = Q_{l,k}(a)$ (where always $k \ne l$) and is called a *partial Q-measure*. The quantity $Q_{k,l}(a)$ is nothing but the binary Q-measure of a with respect to groups G_k and G_l. $TP_{k,l}$ is the number of possible node combinations from groups G_k and G_l. This is $m_k \cdot m_l$ if a does not belong to G_k or G_l. It is $(m_k - 1) \cdot m_l$ if a belongs to G_k and $m_k \cdot (m_l - 1)$ if a belongs to G_l. If $m_k - 1$ or $m_l - 1$ is zero the corresponding TP is taken to be 1 (or any finite number). The exact value plays no role as $p_{g,h}(a) = 0$ in this case. Clearly, for every pair of groups G_k and G_l: $0 \le Q_{k,l}(a) \le 1$. Now $Q_G(a)$ can be written as:

$$Q_G(a) = \frac{1}{C} \sum_{k,l} Q_{k,l} \tag{5}$$

The global Q-measure of a node a always satisfies the inequality $0 \le Q_G(a) \le 1$.
The local Q-measure for node a belonging to group G_d with m_d nodes is defined as:

$$Q_L(a) = \frac{1}{S-1} \sum_{l \ne d} \left(\frac{1}{TP_{d,l}} \sum_{\substack{g \in G_d \\ h \in G_l}} \frac{p_{g,h}(a)}{p_{g,h}} \right) = \frac{1}{S-1} \sum_{l \ne d} Q_{d,l} \tag{6}$$

Note that Eq. (6) uses partial Q-measures as well, but one of the groups must be G_d, the group to which a belongs. Here, $TP_{d,l}$ is equal to $(m_d - 1) \cdot m_l$. Just as for the global Q-measure, the local Q-measure of node a satisfies the inequality $0 \le Q_L(a) \le 1$.

Remark. What happens if the network is unconnected? It was already observed by Freeman (1977) that the definition of betweenness centrality is applicable in this case as well. If g and h belong to different components, there exists no (shortest) path from g to h and therefore $p_{g,h}(a) = p_{g,h} = 0$. This leads to a division by zero in the definition. By convention we set $0/0 = 0$ in this case. Given this convention, one can determine the betweenness centrality, local or global Q-measure for any node in a network, even if it is an isolate (i.e., unconnected to any other node). Indeed: in that case, $C_B(a) = Q_G(a) = Q_L(a) = 0$.

We note that it is possible that a node has local and global Q-measure equal to 1. Moreover, this may happen in a network that is not a star network. An example is provided in Fig. 1.
We can refer to nodes whose Q_L or Q_G value exceeds a threshold as respectively local and global bridges in the network.
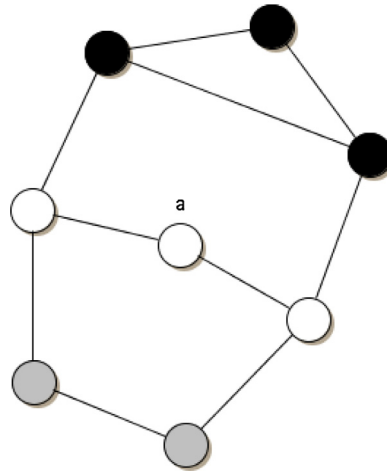
Fig. 2. Example of a node, a, with non-zero betweenness centrality but Q-measures equal to zero.

We also note that when the network is totally unconnected, or when it is completely connected each node has a Q_L and Q_G value equal to zero. This corresponds with the idea of playing the role of a bridge between groups: in the first case there are no bridges whatsoever, and in the second case no bridges are needed.

Q-measures are based on betweenness centrality but take only shortest paths between nodes from different groups into account. Shortest paths between nodes from different groups will be referred to as *inter-group geodesics*. Furthermore, we will distinguish (for a given node a) between external and internal inter-group geodesics. If a belongs to group G, then any geodesic $\gamma_{b,c}(a)$ where $b \in G$ and $c \in (V \setminus G)$ (or vice versa) is an *internal inter-group geodesic* of a. A geodesic $\gamma_{b,c}(a)$ where b, $c \in (V \setminus G)$ is an *external inter-group geodesic* of a. In Fig. 3 (a_2, b_2, c_2) is an external geodesic of b_2; (b_1, b_2, c_1) is an internal geodesic of b_2.

3. Relations between global Q, local Q and betweenness

Since Q_L and Q_G are partly based on the same information (i.e., shortest paths between nodes from different groups), one may expect that a global bridge often plays the role of a local bridge and vice versa. Likewise, it seems reasonable to assume that bridges, and in particular global ones will generally also have a high betweenness centrality. In this section, we examine mathematically which relations are possible between the three measures.

Although $\sum_{l \neq d} Q_{d,l} \leq \sum_{k,l} Q_{k,l}$ we also have that $S - 1 < C = S(S-1)/2$, unless $S = 2$. Hence for $S = 2$, $Q_L(a) = Q_G(a)$ by definition. If $S > 2$ there exist networks such that for some nodes a and b $Q_L(a) < Q_G(a)$ and $Q_L(b) > Q_G(b)$. Examples can be found in Guns and Rousseau (2009).

Theorem 1. If the global or the local Q-measure of a node a is strictly positive, then the betweenness centrality of a is also strictly positive.

Proof. If the (global or local) Q-measure of a is strictly positive, then a is part of at least one geodesic between a pair of nodes from two different groups in the network. Since a belongs to at least one geodesic in the network, by Eq. (2) its betweenness centrality is larger than 0.

We note that the converse of Theorem 1 is not true: it is possible that $C_B(a) > 0$ while $Q_L(a) = Q_G(a) = 0$. In this case, node a is part of one or more geodesics between nodes within its own group. Although it does not play a role in any inter-group geodesic, it may be central to its own group. An example of such a node a is shown in Fig. 2, where black, grey and white indicate nodes belonging to different groups.

We note that if $a \in G_d$

$$\sum_{k,l} Q_{k,l}(a) = \sum_{j \neq d} Q_{d,j}(a) + \sum_{\substack{m,n \\ m \neq d \\ n \neq d}} Q_{m,n}(a). \tag{7}$$

The number $\sum_{\substack{m,n \\ m \neq d \\ n \neq d}} Q_{m,n}(a)$ can be considered as the fractionally counted number of *external inter-group geodesics* of a. As the left-hand side of Eq. (7) leads to the global Q measure of node a, and the first term of the right-hand to the local

Q-measure we introduce an external Q-measure of a based on the second term of the right-hand side. First we note that the *external inter-group geodesics* connect $C - (S - 1) = ((S \cdot (S - 1))/2) - (S - 1) = ((S - 1) \cdot (S - 2))/2$ groups.

Definition. External Q-measure of a node a

The external Q-measure of node $a \in G_d$, denoted as $Q_E(a)$ is defined as:

$$Q_E(a) = \frac{2}{(S - 1) \cdot (S - 2)} \sum_{\substack{m,n \\ m \neq d \\ n \neq d}} Q_{m \cdot n}(a)$$

Clearly $0 \leq Q_E(a) \leq 1$.

Next we introduce the following definition.

Definition. A real-valued function F defined on a set X is said to be a convex combination of real functions G and H if there exists a number t, $0 \leq t \leq 1$, such that $F = t \cdot G + (1 - t) \cdot H$. Conversely, if F, G and H are given and it is possible to find a real number t, $0 \leq t \leq 1$, such that $F = t \cdot G + (1 - t) \cdot H$ then one has obtained a convex decomposition of F by means of G and H.

Theorem 2. If a belongs to G_d then we have the following convex decomposition of $Q_G(a)$

$$Q_G(a) = \frac{2}{S} Q_L(a) + \frac{S - 2}{S} Q_E(a) \tag{8}$$

Proof. From Eq. (4), the definition of the global Q-measure, where $Q_{k,l}(a)$ is the partial Q-measure of a with respect to groups G_k and G_l we see that:

$$Q_G(a) = \frac{1}{C} \sum_{k,l} Q_{k,l}(a) = \frac{1}{C} \left[\sum_{j \neq d} Q_{d,j}(a) + \sum_{\substack{m,n \\ m \neq d \\ n \neq d}} Q_{m,n}(a) \right]$$

$$= \frac{1}{C} \sum_{j \neq d} Q_{d,j}(a) + \frac{1}{C} \frac{(S - 1) \cdot (S - 2)}{2} \frac{2}{(S - 1) \cdot (S - 2)} \sum_{\substack{m,n \\ m \neq d \\ n \neq d}} Q_{m,n}(a) = \frac{1}{C}(S - 1)Q_L(a) + \frac{1}{C} \frac{(S - 1) \cdot (S - 2)}{2} Q_E(a)$$

In other words,

$$Q_G(a) = \frac{2}{S} Q_L(a) + \frac{(S - 2)}{S} Q_E(a). \tag{8}$$

This proves Theorem 2.

Consider the example in Fig. 3, previously used by Guns and Rousseau (2009). Counting all inter-group geodesics we found

$$Q_G(b_2) = \frac{1}{3} \left(\frac{1}{6} + \frac{19}{27} + \frac{3}{6} \right) = \frac{37}{81}.$$

Indeed, the same result is obtained using the convex decomposition (8):

$$Q_G(b_2) = \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{19}{27} = \frac{37}{81}$$

Guns and Rousseau (2009) also considered two 'countries' unconnected to each other, but connected to a 'single-city state' – Fig. 4 provides an example. This single-city state always has Q_L equal to zero and Q_G equal to 1/3. In this case the external inter-group contribution is equal to the (complete) global Q-value.

Corollary. If $Q_L(a) > 0$ or if $Q_E(a) > 0$, then $Q_G(a) > 0$.

We observe that it is possible that a node's local Q-measure is equal to zero, while its global Q-measure is larger than zero. Specifically, as can easily be seen from the proof, if $Q_L(a) = 0$, then $Q_G(a) = ((S - 2)/S)Q_E(a) \geq 0$. In this case, node a is not part of any internal inter-group geodesic, but it may be part of an external inter-group geodesic as illustrated above for the single-city state.
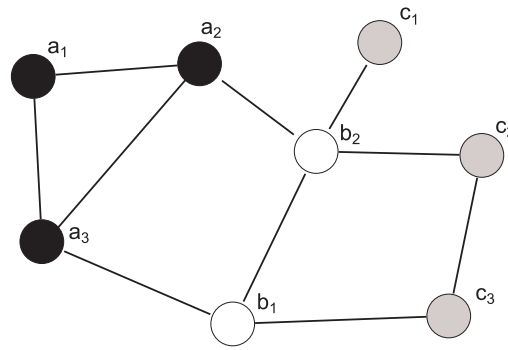
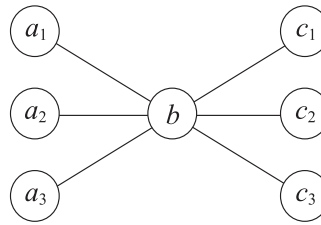**Fig. 3.** Example network with three groups (black, white and grey).

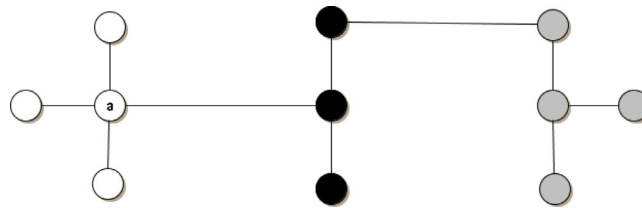**Fig. 4.** Network with three groups $\{a_1, a_2, a_3\}$, $\{b\}$ and $\{c_1, c_2, c_3\}$.

Fig. 5. Maximum difference between global and local Q-measure for node a in a three group network.

Proposition 1.

$$|Q_L(a) - Q_G(a)| \le \frac{S - 2}{S}$$

Proof. This follows immediately from Eq. (8) (the convex decomposition) and the inequality shown in Appendix A.

Corollary.

If $Q_L(a) = 0$ then $Q_G(a)$ is not equal to 1.
If $Q_L(a) = 1$ then $Q_G(a)$ is not equal to 0.
If $Q_G(a) = 0$ then $Q_L(a)$ is not equal to 1.
If $Q_G(a) = 1$ then $Q_L(a)$ is not equal to 0.

It may seem that the inequality in Proposition 1 is rather coarse. Yet, it is the best possible. Indeed, consider the example shown in Fig. 4, which illustrates an extreme case: $Q_L(b) = 0$ but all partial Q-measures not involving group $\{b\}$ are equal to 1. In this case, $Q_G(b) = (S - 2)/S$ and $Q_E(b) = 1$. We note for further reference that if a group is a singleton $G = \{b\}$ then the local Q-value of b is always zero.

Proposition 2. If a group G is a singleton: $G = \{b\}$ then $Q_L(b) = 0$.

If $Q_L(a) = 1$, then by (8) $Q_G(a) = (2/S) + ((S - 2)/S)Q_E(a)$. If node a is not part of an external inter-group geodesic, then $Q_G(a) = 2/S$. Again, we find that the difference (in this case, $Q_L(a)$ is the largest) between a global and local Q-measure is equal to $1 - (2/S)$. Such a situation is illustrated in Fig. 5.

We see that the maximum difference between the local and the global Q-measure increases with S, the number of groups.

4. When are the maximum possible values of Q-measures obtained?

In this section we characterize nodes a for which $Q_G(a) = 1$, $Q_L(a) = 1$ or $Q_E(a) = 1$. As the node in a singleton group cannot have a global Q-measure equal to 1, we assume in this section that node a's group is not a singleton group.

Lemma. If a is a node in a network partitioned in two or more groups then the following two expressions are equivalent:

(1) $Q_G(a) = 1$.
(2) $Q_L(a) = 1$ and $Q_E(a) = 1$

This follows immediately from the convex decomposition of $Q_G(a)$.

Consequently, we will first investigate when $Q_L(a) = 1$ and when $Q_E(a) = 1$.

Proposition 3. If node a belongs to group G_d in a network partitioned in two or more groups then the following three expressions are equivalent:

(1) $Q_L(a) = 1$.
(2) For each node pair $(g,h) \in (G_d, G_l)$, $l \neq d$ (g different from a), g and h are connected and node a belongs to each shortest path between g and h.
(3) The network is connected and node a is part of each inter-group link between G_d and any other group in the network.

Proof. We first show that (1) and (2) are equivalent.
We note that for $a \in G_d$ $Q_L(a) = 1$ if and only if

$$\sum_{l \neq d} \frac{1}{TP_{d,l}} \left(\sum_{\substack{g \in G_d \\ h \in G_l}} \frac{p_{g,h}(a)}{p_{g,h}} \right) = S - 1$$

This happens if and only if, for each group $G_l (l \neq d)$:

$$\frac{1}{TP_{d,l}} \sum_{\substack{g \in G_d \\ h \in G_l}} \frac{p_{g,h}(a)}{p_{g,h}} = 1$$

This means that for every node $g \in G_d$ (g different from a) and $h \in G_l$ there exists a shortest path (at least one) and node a belongs to each of these shortest paths.
Next, we show that (3) implies (2)
Consider a pair of nodes g and h with $g \in G_d, g \neq a$, and h belonging to another group and any shortest path between them (this exists as the network is connected). Such a shortest path must include an inter-group link. Since a is situated on each of them, a also belongs to each shortest path.
Finally, we show that (2) implies (3)
Every inter-group link between G_d and another group is by definition a shortest path between a node in G_d and one in another group. By (2) we know that such a path contains node a.
Now we prove that the network is connected by considering four cases. Consider two nodes g_0 and h_0 (both different from a). If $g_0 \neq a$ belongs to G_d and h_0 belongs to another group then there exists a shortest path between g_0 and h_0: hence these nodes are connected. If g_0 and h_0 do not belong to G_d, then we consider a node b in G_d, different from a. Then g_0 is connected to b and b is connected to h_0, hence g_0 and h_0 are connected. Consider now node a and any other node h_0 outside G_d. Again we use node b in G_d to obtain a shortest path between h_0 and b via a. Hence a is connected to h_0. Finally, we consider nodes a and b in G_d. As the network contains at least two groups there exists a node h_1 outside G_d. There also exists a shortest path between b and h_1 via a, hence a is connected to b. Taken together, these four cases prove that the whole network is a connected network.
This proves Proposition 3.

Proposition 4. If node a belongs to group G_d in a network partitioned in two or more groups then the following three expressions are equivalent:

(1) $Q_E(a) = 1$.
(2) For each pair $(g,h) \in (G_k, G_l)$, $l \neq d$, $k \neq d$, g and h are connected and node a belongs to each shortest path between g and h.
(3) Node a is situated on each inter-group link between any two groups, each different from G_d, in the network.

Proof.   We first show that (1) and (2) are equivalent.

We note that for $a \in G_d$ $Q_E(a) = 1$ if and only if

$$\sum_{\substack{k,l \\ k \neq d \\ l \neq d}} Q_{k,l}(a) = \frac{(S-1) \cdot (S-2)}{2}$$

This happens if and only if, for every $k,l \neq d$:

$$Q_{k,l}(a) = \frac{1}{TP_{k,l}} \sum_{\substack{g \in G_k \\ h \in G_l}} \frac{p_{g,h}(a)}{p_{g,h}} = 1$$

This means that for every pair of nodes $(g,h) \in (G_k, G_l)$, $k,l \neq d$ there exists a shortest path (at least one) and node a belongs to each of these shortest paths.

Now, we show that (3) implies (2).

Consider a pair of nodes g and h, $(g,h) \in (G_k, G_l)$ $k,l \neq d$ and any shortest path between them (this exists as the network is connected). Such a shortest path must include an inter-group link and a is situated on each of them.

Finally we show that (2) implies (3).

Every inter-group link is by definition a shortest path between two nodes from different groups. By (2) we know that such a path contains node a.

This proves Proposition 4.

We note that $Q_E(a) = 1$ does not imply that the network is connected. If we expand in Fig. 4, the middle group to $\{b, b_0\}$, where b_0 is disconnected from all nodes then still $Q_E(b) = 1$.

Bringing the results of the previous lemma, Propositions 3 and 4 together leads to the following Theorem 3.

Theorem 3.   If a is a node in a network partitioned in two or more groups then the following three expressions are equivalent:

(1) $Q_G(a) = 1$.
(2) For each pair of nodes $(g,h) \in (G_k, G_l)$, $k \neq l$, both different from a, there exists (at least one) shortest path between g and h such that a belongs to all shortest paths between g and h.
(3) The network is connected and node a is situated on each inter-group link in the network.

Corollary 1.   If $Q_G(a) = 1$ then a is directly connected to at least one node of each group.

Proof.   Node a is not an isolate as the network is connected. Consider a group G_j different from a's group and a node h in this group. Consider a shortest path between a and h. This path must contain an inter-group link. As a is situated on each inter-group link, a is directly connected to G_j.

It is possible that $Q_L(a) = 1$ while $Q_G(a)$ is not. An example is provided by the following network: $G_1 = \{x_1, x_2, x_3\}$, $G_2 = \{y_1, y_2\}$ and $G_3 = \{z_1, z_2\}$ as a line graph:

$$x_1 - x_2 - x_3 - y_1 - y_2 - z_1 - z_2$$

Here $Q_L(x_3) = 1$, but $Q_G(x_3) = 2/3 < 1$.

Corollary 2.   If a belongs to group G_d and $Q_L(a) = 1$ then a is directly connected to at least one node in G_d.

Proof.   Assume that $G_d = \{a, b, \ldots, v\}$ and that a is not connected to any of the others. Then at least one of them is connected to an element that belongs to another group, otherwise the set $\{b, \ldots, v\}$ would be a component (or more than one) on its own, which is not possible by Proposition 3. But this means that there exists an inter-group link that does not involve a, which is again not possible by Proposition 3. This proves this corollary.

Finally, we briefly look into the question when $Q_G(a) < Q_L(a)$ or vice versa. We recall that if $S = 2$, $Q_L(a) = Q_G(a)$ by definition. We now assume that $S > 2$. By (8), we have that

$$Q_G(a) - Q_L(a) = \frac{2}{S} Q_L(a) + \frac{S-2}{S} Q_E(a) - Q_L(a) = \frac{S-2}{S} Q_E(a) - \frac{S-2}{S} Q_L(a) = \frac{S-2}{S}(Q_E(a) - Q_L(a))$$

Hence: $Q_G(a) > Q_L(a)$ if and only if $Q_E(a) > Q_L(a)$. This shows that the relation between $Q_G(a)$ and $Q_L(a)$ depends on the number of external inter-group geodesics, i.e., how often a forms part of geodesics between nodes from two other groups. $Q_G(a)$ being greater than $Q_L(a)$ thus indicates that a primarily maintains multilateral relations with nodes from many different groups.

5. Conclusion

In this contribution we studied group interaction and the brokerage role of nodes in undirected unweighted networks based on Q-measures. We introduced an external Q-measure to complement the previously defined local and global Q-measure and investigated the meaning of numerical values obtained by these measures. In particular we obtained the global Q-measure as a convex combination of the local and the external Q. We further investigated when a node can have a Q-value equal to one (the largest possible value).

Acknowledgement

Appendix A.

If $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and z is a convex combination of x and y, which means that $z = t.x + (1 - t).y$, with $0 \leq t \leq 1$, then $|z - x| \leq (1 - t)$.

Proof. We first show that $|y - x| \leq 1$. Indeed, without loss of generality we may assume that $x \leq y$, leading to: $0 \leq y - x \leq 1 - x \leq 1$.

Then we see that $|z - x| = |t \cdot x + (1 - t) \cdot y - x| = |(t - 1) \cdot x + (1 - t) \cdot y| = (1 - t) \cdot |y - x| \leq (1 - t)$.

This proves the inequality.

References

Anthonisse, J. M. (1971). *The rush in a directed graph*. Technical Report BN 9/71, Stichting Mathematisch Centrum.

Brandes, U. (2008). On variants of shortest-path betweenness centrality and their generic computation. *Social Networks*, *30*(2), 136–145.

Chen, L. X., & Rousseau, R. (2008). Q-measures for binary divided networks: Bridges between German and English institutes in publications of the Journal of Fluid Mechanics. *Scientometrics*, *74*(1), 57–69.

Everett, M. G., & Borgatti, S. P. (1999). The centrality of groups and classes. *Journal of Mathematical Sociology*, *23*(3), 181–202.

Flom, P. L., Friedman, S. R., Strauss, S., & Neaigus, A. (2004). A new measure of linkage between two sub-networks. *Connections*, *26*(1), 62–70.

Franceschet, M. (2012). The large-scale structure of journal citation networks. *Journal of the American Society for Information Science and Technology*, *63*(4), 837–842.

Freeman, L. C. (1977). A set of measures of centrality based upon betweenness. *Sociometry*, *40*(1), 35–41.

Girvan, M., & Newman, M. E. J. (2002). Community structure in social and biological networks. *Proceedings of the National Academy of Sciences of the USA*, *99*(12), 7821–7826.

Goh, K. I., Cusick, M. E., Valle, D., Childs, B., Vidal, M., & Barabási, A. L. (2007). The human disease network. *Proceedings of the National Academy of Sciences*, *104*(21), 8685–8690.

Guns, R., & Liu, Y. X. (2010). Scientometric research in China in the context of international collaboration. *Geomatics and Information Science of Wuhan University*, *35*(ICSUE 2010 special), 112–115.

Guns, R., Liu, Y. X., & Mahbuba, D. (2011). Q-measures and betweenness centrality in a collaboration network: A case study of the field of informetrics. *Scientometrics*, *87*(1), 133–147.

Guns, R., & Rousseau, R. (2009). Gauging the bridging function of nodes in a network: Q-measures for networks with a finite number of subgroups. In B. Larsen, & J. Leta (Eds.), *Proceedings of ISSI 2009 – The 12th international conference on scientometrics and informetrics* (pp. 131–142).

Leydesdorff, L. (2007). Betweenness centrality as an indicator of the interdisciplinarity of scientific journals. *Journal of the American Society for Information Science and Technology*, *58*(9), 1303–1319.

Liu, Y. X., Rafols, I., & Rousseau, R. (2012). A framework for knowledge integration and diffusion. *Journal of Documentation*, *68*(1), 31–44.

Liu, Y. X., Rousseau, R., & Guns, R. (2013). A layered framework to study collaboration as a form of knowledge sharing and diffusion. *Journal of Informetrics*, *7*(2), 651–664.

Newman, M. E. J. (2003). The structure and function of complex networks. *SIAM Review*, *45*(2), 167–256.

Otte, E., & Rousseau, R. (2002). Social network analysis: A powerful strategy, also for the information sciences. *Journal of Information Science*, *28*(6), 441–453.

Risse, T. (2000). "Let's argue!": Communicative action in world politics. *International Organization*, *54*(1), 1–39.

Rousseau, R. (2005). Q-measures for binary divided networks: An investigation within the field of informetrics. *Proceedings of the 68th ASIST Conference*, *42*, 675–696.

Rousseau, R., Liu, Y., & Ye, F. Y. (2012). A preliminary investigation on diffusion through a layered system. *Journal of Informetrics*, *6*(2), 177–191.

Rousseau, R., & Zhang, L. (2008). Betweenness centrality and Q-measures in directed valued networks. *Scientometrics*, *75*(3), 575–590.

Wasserman, S., & Faust, K. (1994). *Social network analysis: Methods and applications*. Cambridge University Press.

Yan, E., & Ding, Y. (2009). Applying centrality measures to impact analysis: A coauthorship network analysis. *Journal of the American Society for Information Science and Technology*, *60*(10), 2107–2118.