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Kurt Weichselberger's contribution to imprecise probabilities and statistical inference ☆

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ABSTRACT

Kurt Weichselberger, one of the influential senior members of the imprecise probability community, passed away on February 7, 2016. Almost throughout his entire academic life the major focus of his research interests has been on the foundations of probability and statistics. The present article is a first attempt to trace back chronologically the development of Weichselberger's work on interval probability and his symmetric theory of logical probability based on it, aiming at a new framework for statistical inference. We also try to work out the intellectual background of his different projects together with some close links between them.

1. Introduction

Kurt Weichselberger, who passed away on February 7, 2016, has been "a man of the first hour" of the ISIPTA meetings, perceiving them as the natural place to discuss the foundations of probability. He enthusiastically participated in the first six ISIPTAs, from the 1999 Ghent symposium to the Durham meeting in 2009, contributing several papers, a tutorial in 2005 and a special session in 2009. From the mid sixties of the last century onwards, the foundations of statistics and probability have always been Weichselberger's great passion – although he had worked on a variety of different topics,[1] and had been intensively engaged in academic self-administration and societies. By this engagement he had a lasting impact on the academic self-organization of German academics in general and the discipline of statistics in particular.

This paper is a first attempt to trace back fundamental aspects of Weichselberger's ideas as well as their links to his challenging research program. Our work is embedded into the *HiStaLMU* project (History of Statistics at LMU Munich). Among other activities, its members interview former leading personalities of the Department of Statistics by using methods of oral history [7] and build up an archive around Kurt Weichselberger's estate.[2]

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[1] The work on applied statistics includes among others research on survey and census methodology (e.g. Weichselberger [48]), regional price indices [69], and time series [49], see also Section 4.4.

[2] See also the workshop in March 2016 (https://statsoz-neu.userweb.mwn.de/research/ws_historystatistik_2016/index.html); last access April 4, 2018.

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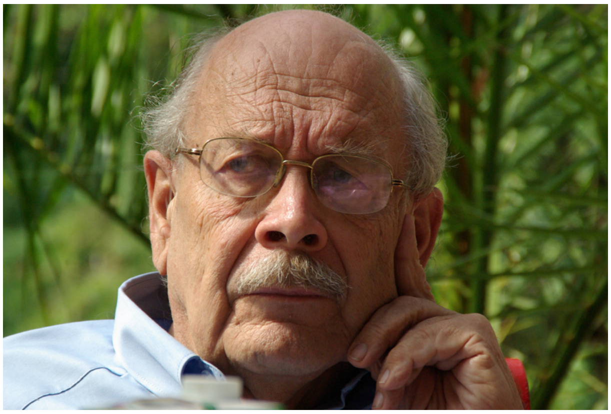

Fig. 1. Kurt Weichselberger (photo kindly provided by Kurt Weichselberger's family).

This paper is structured as follows. After a brief biographic sketch (Section 2), Section 3 briefly looks at those activities in academic self-administration that are motivated by Weichselberger's understanding of statistics as a general methodology. In the main part, focusing on Weichselberger's foundational work, the structure of presentation in this paper is chronologically. We distinguish four main periods: the first intensive research on logical probability (see Section 4), the work on probability intervals in the context of modelling uncertain expert knowledge (Section 5), the axiomatic foundations of an interpretation independent theory of interval probability in his book *Elementare Grundbegriffe...* and its direct predecessors (Section 6), and eventually the aim to synthesize the previous results towards the *symmetric theory of logical probability* (Section 7) as a general framework for statistical inference. Section 9 concludes.

2. A short biographic sketch

In this section we briefly summarize the main stages of Weichselberger's career.[3]

Kurt Weichselberger (see also Fig. 1) was born in Vienna on April 13, 1929. He studied mathematics there, and worked from 1951 to 1953 as a junior assistant (*Wissenschaftliche Hilfskraft*) to Leopold Schmetterer and Wilhelm Winkler at the Department of Statistics in Vienna. In 1953 he earned his PhD (*Dr. phil.*) for a thesis on Bernstein polynomial approximation supervised by Johann Radon [45]. Until 1960 he was employed at a social research institute in Dortmund. Afterwards Weichselberger joined Johann Pfanzagl's chair in Cologne, where he received his *Habilitation* in 1962 with a thesis on controlling census results [48].

From 1963 to 1969 Weichselberger held the chair in statistics at the Technische Universität Berlin. In 1967 he was elected rector of this university and substantially contributed to the by then vivid public debate about the role of education and scientists in the modern society with some visionary ideas, anticipating many aspects of the recent discussion in Germany. This includes the comparably high age of German students, the squeezing of the German mass universities between the demands for a quick preparation for the labor market versus a long-term preparation for research and a continuously changing society, the self-referential character of examinations, and the self-production of social status through the German education system (see, e.g., Weichselberger [52], or also Rüger [31, pp. 6–10]).

From April 1969 on, for almost 50 years, Weichselberger has been a member of the Ludwig–Maximilians–Universität München (LMU Munich), where he also was the driving force in establishing an autonomous Department of Statistics and Philosophy of Science (see also Section 4.3). From 1997 on, Weichselberger continued his research activities as a professor emeritus. On February 7, 2016, he passed away in his house in Grafing among close family.

3. Statistics as a general methodology: institutional activities

Weichselberger's research can not be understood without studying his deep engagement in academic self-administration. It had been driven by his understanding of statistics as a methodology that is independent of concrete application fields, and this fact had to be expressed by the institutional alignment of statistics. Indeed, in Germany, Weichselberger has been perceived as one of the major driving forces for the institutionalization of statistics as a discipline of its own, locally at LMU Munich, as well as in the German university system.

In his first years in Munich, Weichselberger had worked intensively on emancipating statistics from political and economic sciences, and thus in particular from its fixation on any specific area of application, but also without reducing it

[3] For more details see in particular Rüger [31] and Rüger's obituary [32]. Many students and academic companions until the mid nineties are assembled in the Festschrift edited by Rinne, Rüger, and Strecker [30].

to a sub-discipline of mathematics. Institutionally,[4] this became most visible at LMU Munich in the early seventies, when the faculty structure was reorganized. Weichselberger convinced his colleagues not to join one of the newly arising Faculties of Management Science, Economics or Social Sciences, but to emphasize the genuine general character of statistics by constituting the Faculty of Philosophy, Philosophy of Science and Statistics. Within that faculty, in 1974 the Department of Statistics and Philosophy of Science was founded, comprising the three chairs in statistics (and econometrics) and the chair in philosophy of science held by Wolfgang Stegmüller (see also Section 4.3.1).

In the German Statistical Society Weichselberger had held the chair of the Education Section for more than 10 years (1969–1981) and had been a member of the society's Executive Board. In that role, he had a strong influence on the embodiment of the German wide curricula of that programmes. The question about the position and foundations of statistics – statistics in the first line as social, economic and also official statistics versus statistics as a general methodology, somewhat isolated from the concrete object of investigation – had also been the topic of an intensive controversy in the German Statistical Society. This controversy has been much more intensive in Germany than in Anglo–Saxon countries. To some extent lasting until today, it indeed has been even constitutive for the development of statistics at German universities (see, e.g., Härdle and Vogt [22] and Fischer and Seising [19], in particular Vogt [38]). Particularly, it was reflected in ongoing discussions about the role economic and official statistics should play in curricula of study programmes in management science and economics (see, especially, the debate reflected, and renewed, in von der Lippe and Schmerbach [39]). As the Chairman of the Education Section, combined with his activities for the association of German universities offering study programmes in economic and social sciences (*Wirtschafts- und Sozialwissenschaftlicher Fakultätentag*), Weichselberger stood in the centre of this debate. Although he had been working on social and official statistics for quite some time, he had taken a clear position in favour of statistics as a general methodology. This position eventually became widely accepted, and teaching of general methods of descriptive and inductive statistics became the standard in Germany, at the cost of economic statistics.[5]

Weichselberger also had a strong impact on the development of the first study programmes in statistics as a major subject in Germany, which eventually were established in the seventies in Dortmund and in Munich. In Munich, Weichselberger was, of course, also intensively involved in the concrete implementation of the programme.

4. Logical probability I

In this section we look at Weichselberger's work on logical probability from the sixties to the early eighties. This work is hardly documented. The only available, yet quite short, publication is Weichselberger's inaugural speech as rector in Berlin (available in printed form as Weichselberger [51]), where he set out for his great scientific mission and passion: the development of a new theory of statistical inference, putting Fisher's fiducial argument back on its feet and substantially extending it. Without doubt, many important ideas of imprecise probabilities and statistical inference based on it are anticipated and already sketched in this programmatic speech. We document this work in Section 4.2 by discussing and quoting some of the most important parts, translating them from the German original text.

This documentation is prepared by Section 4.1, where we try to give a first, quite simplified, summary of Weichselberger's concept of logical probability. In Section 4.3 we report our first insights into the work in the first two decades in Munich, including its intellectual background.

4.1. Logical probability as a two-place concept

Weichselberger emphasises that a general inference theory has to be built on what he calls *logical probability*, assigning probabilities to inferences/reasoning between propositions, typically from a set $\mathfrak{P}$ of premises to a set $\mathfrak{C}$ of conclusions. Weichselberger rigorously understood such a logical probability as a two-place function[6] $P(\mathfrak{C}||\mathfrak{P})$, to be read as the "probability of $\mathfrak{C}$ under the hypothesis $\mathfrak{P}$", i.e. the probability with which $\mathfrak{C}$ can be concluded from $\mathfrak{P}$.[7] This concept of probability describes probabilistic deduction and is, by construction, non-subjective.

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[4] See also Krem [27, Chapter 3] and the interviews with (former) department members (Hans Schneeweiss, who held the Chair in Econometrics and Statistics, Christina Schneider, a former assistant to Weichselberger, and Christa Jürgensonn, a technical assistant at the department,) within the HiStaLMU project [7].

[5] In a recent interview, Heinz Grohmann, honorary chairman of the German Statistical Society remembers: "In the 1950s there was a dispute between the 'Munich School' and the 'Frankfurt School' [...] In Munich the statisticians considered statistics as a general methodology, independent from the field of application. In Frankfurt they emphasized that economic and social sciences not only make formal but also material demands on statistics. In fact, on closer inspection there is no contradiction between the two. However, pure formally developed statistics got widely accepted.

A basic course of two terms in statistics which was specialized in probability theory, theory of estimation and testing theory became standard. Courses on economic statistics remained an exception. Regrettably!

[...I]n general, [...], statistics in the sense of Weichselberger won the day. Although almost all chairs in statistics were established in the fields of economics, only little attention has been paid to the big economic and social problems of the time, e.g. demographic change, social inequality, the globalization or the energy transition." (Krämer [26, p. 272] [translated by the authors]).

[6] In this respect Weichselberger may have been influenced by the work of Carnap and Stegmüller, see Section 4.3.1.

[7] This naturally reflects the etymological basis of the word probability as *prove–ability*, as well as the constituents of the corresponding German word *Wahr–schein–lich–keit*, characterizing the extent to which something seems to be true.

The crucial issue to derive a framework for statistical inference from this is the inversion[8] of $P(\mathfrak{C}||\mathfrak{P})$, transforming it into $P(\mathfrak{P}||\mathfrak{C})$. This inversion connects probabilistic deduction with induction, and indeed frames statistical inference: If a statistical model is considered, describing the probabilistic deduction of the probability of every event A given a parameter value ϑ in some parameter space Θ, and formulated as a two-place probability $P(A||\{\vartheta\})$, the parameter value ϑ becomes a premise and the event A takes the role of a set of conclusions. After having seen the data A^*, the inversion provides us with the two-place probability $P(\{\vartheta\}||A^*)$, which is nothing less than the non-subjective probability with which we can conclude from the data A^* to every specific parameter value $\{\vartheta\}$. We then have a kind of objective posteriori distribution of the parameter given the data that is produced without any specification of a prior, allowing to describe the degree of support data give to statistical models.

Weichselberger calls his theory "symmetric theory", because the role of $\mathfrak{P}$ and $\mathfrak{C}$, as the arguments of the two-place probabilities, are interchanged here, and therefore they show a kind of symmetry.

4.2. Setting the scene: Weichselberger's inaugural speech as a rector in Berlin

In his inaugural speech Weichselberger initially reviews some aspects of the history of probability and statistics, discusses the objectivistic concept in some details, including major lines of criticism on it. The last part of the speech is devoted to sketching some basic ideas towards a general theory of statistics, based on a substantially extended notion of probability. Weichselberger explicates:

> *[... W]e are challenged with the task to reconceptualise the foundations of probability. The question is whether we can make progress towards a broader concept designation without losing key benefits of the previous – objectivistic – concept. For that matter we have to decide which properties of the objectivistic probability concept we consider to be essential.*
>
> *[...The] two essential properties of Kolmogorov's probability concept that consequently may not be waived [are the following].*
>
> 1. *The embedding into modern mathematics.*
>   *This needs to remain ensured by guaranteeing the mathematical properties of the concept by a consistent system of axioms.*
> 2. *The possibility of the frequency interpretation of probabilities.*
>   *This presents to date the only known mindset that enables an explanation of the concept and thus guarantees that the ideas of different persons on the meaning of probability can be adapted.*
>
> *Taking these issues into account we are challenged with the task as follows. We have to develop a system of axioms that*
>
> 1. *includes Kolmogorov's system of axioms as a special case;*
> 2. *associates probabilities not with events but with inferences from premises to conclusions;*
> 3. *enables the frequency interpretation of the probability concept;*
> 4. *enables probability propositions in both directions in cases in which the Fisher theory and the Neyman–Pearson theory yield the same results; for example in the case of a sample from a population, it associates a probability with the inference from the population to the sample as well as with the inference from the sample to the population.* (Weichselberger [51, pp. 46–47] [translated by the authors])

Weichselberger was already very clear about the fact that such a theory necessarily has to go beyond the restrictions precise probabilities imply, and therefore continues:

> *As in many cases in the history of science it is shown also here that — as a form of compensation for desired benefits – we have to abandon a 'habit of thinking' (*Denkgewohnheit*). In the present case this is the habit of thinking that the probability is always a number. We must instead allow sets of numbers – say the interval between 0.2 and 0.3 – to act as the probability of the inference from the proposition B to the proposition A. However, we continue to demand that the set of numbers lies in the interval between 0 and 1.*
>
> *This extension of the probability concept from a number to a set of numbers is encouraged as soon as we try to formalize Fisher's fiducial probability. Therefore a similar approach has already been taken by the American Henry Kyburg Jr. in his works in the years 1961 to 1964. However, Kyburg's concept is inconsistent at a decisive point, and, to his own statements, it does not lead to useful results in detail. His view is mainly of philosophical and not of mathematical nature.*
>
> *In fact, the definition of probability as a set of numbers – normally a single number or an interval – leads to mathematical problems. We need a system of calculation rules for algebraic operations with such sets. This prompts us to similar considerations as the systematization of calculations with inexact or error-prone quantities: one could call it a 'theory of tolerance space' (*Spielraum-Theorie*) because we associate tolerance spaces instead of individual numbers. I think that it is possible that this view may give rise to interesting inner mathematical questions.* (Weichselberger [51, p. 47] [translated by the authors])

4.3. Further Early work on logical probability in Munich

During this period, Weichselberger had accepted an offer from LMU Munich for a newly installed chair for *Special Topics in Statistics (Spezialgebiete der Statistik)*. In Munich he was strongly engaged in changing the institutional alignment of statistics

[8] Weichselberger uses the term "Rückschluss" for it, which could be literally translated as "backward conclusion".

within the university. In 1974, the Institute for Statistics and Philosophy of Science was founded as a member of the new Faculty of Philosophy, Philosophy of Science and Statistics (see also Section 3). Weichselberger has stayed in intensive intellectual contact with his colleagues from philosophy all the time. Clearly, there have been common scientific interests in particular with Wolfgang Stegmüller, who held the Chair in Philosophy of Science and was co-founder of the Department of Statistics and Philosophy of Science.

4.3.1. Stegmüller and his work on Carnap's concept of probability

Stegmüller was appointed professor for philosophy, logic, and philosophy of science at the LMU in 1958, and he remained in this position until his retirement in 1990. His research covered, among other topics, subjective probability and Carnap's concept of logical probability.[9] In 1959, Stegmüller and Carnap published the German book "Induktive Logik und Wahrscheinlichkeit" ["Inductive Logic and Probability"] [12]. It was originally planned as a German translation of the small booklet "The Nature and Application of Inductive Logic" [11], which consists of six sections from Carnap's book "Logical Foundations of Probability", already published in 1950 [10]. However, because Stegmüller wrote some new chapters and a short introduction on Carnap's view on inductive Logic, this German book appeared as a self-contained work of these two authors. In his introduction, Stegmüller emphasized that in Carnap's view the term *probability* is ambiguous. Depending on the meaning that is used, different methods of explication of this concept are required. Carnap called the probability within the meaning of logic *inductive probability*, and he distinguished this concept from the probability within the meaning of statistics (*statistical probability*). Carnap denoted inductive probability as *probability*$_1$ and statistical probability as *probability*$_2$ (Carnap and Stegmüller [12, p. 5]). Probability$_1$ and probability$_2$ are different concepts, however, both concepts have logical character in Carnap's view, Stegmüller summarized: "The two concepts of probability$_1$ and probability$_2$ are similar because they are functions of two arguments with real values in the interval 0 to 1. However, they are different in two other views. The two arguments of probability$_1$ are propositions (hypothesis and data); the two arguments of probability$_2$ are properties or classes. An elementary probability$_1$-proposition is always logically true or logically false, and therefore it does not assert a matter of fact; at the contrary, an elementary probability$_2$-proposition asserts a matter of fact and is therefore empirical." (Carnap and Stegmüller [12, p. 28f] [translated by the authors])

4.3.2. Fragments of a book prepared by Weichselberger

In the first years in Munich, Weichselberger worked intensively on a book in German language, comparing concepts of probability and further developing his concept of logical probability. However, it remained incomplete, and Weichselberger had stopped working on it a long time ago. We found in his home fragments of almost 200 pages of the first chapter in a rather advanced stage. Up to now, it is unclear whether further material exists and even, since there is also no original table of contents available so far, whether it existed at some point. We cite this material as Weichselberger [70] and use the title of the first chapter "Wahrscheinlichkeitstheorien im Vergleich" ("Theories of Probability in Comparison" [translated by the authors]).

The material is not yet evaluated and studied in detail. Moreover, we still search for further material. Therefore we restrict ourselves here to the following brief overview.

Weichselberger [70] discusses the three major concepts of probability, the objective concept (Section 1.2 of the manuscript), the subjective concept (Section 1.3) and the logical concept (Section 1.4). The latter part is by far the largest one. There, Weichselberger reviews the historical development (Section 1.4.1), but also tries to develop his concept of logical probability in detail. Section 1.4.2 is entitled "Foundations of a consequential theory of the logical concept of probability" [translated by the authors], Section 1.4.3 elaborates the concept of tolerance spaces (Spielräume[10]). Section 1.4.4, 1.4.6 and 1.4.7 propose several axioms for unconditional and conditional logical probability; Section 1.4.5 is missing.

Weichselberger ascribes the concept of logical probability to the German philosopher Gottfried Wilhelm Leibniz who used in his juridical PhD thesis "De Conditionibus" ["On Conditions"] a first sketch of a quantitative concept for probability in juristic problems. Without referencing to Boole [8], see also Hailperin [21], Weichselberger claims that there is no tradition of a formalized quantitative logical probability concept before the 20th century, with the works of John Maynard Keynes, Harold Jeffreys and Rudolf Carnap, who considered logical probability from the philosophical perspective. Thus, he started his study with this diagnosis that, for the concept of logical probability "[...] there has hitherto been no formalizable quantification that is self-contained, i.e. not falling back on methods which are typical for the other probability concepts." And he continued: "To close this gap is the goal of the study at hand." (Weichselberger [70, first page of Section 1.4] [translated by the authors])

Following Savage's description of the three basic concepts of probability [33], Weichselberger recapitulates: "Every theory of the logical probability concept forms an attempt to express rules for a *non-deductive derivation of propositions from others and to evaluate them using a quantification*." (Weichselberger [70, first page of Section 1.4] [translated by the authors; emphasis by Weichselberger]) Therewith, logical probability theory is, according to Weichselberger (ibid.), closely related to what is called *inductive logic* in the field of philosophy with important protagonists in the modern age, like David Hume and John

[9] To which extent a concrete co-operation in research has taken place between Weichselberger and Stegmüller is still an open question, which shall be studied further within the HiStaLMU project.

[10] Compare the last paragraph of Section 4.2 here, documenting Weichselberger [51].

Stuart Mill. However, these philosophers had not been interested in formalization and quantification of inductive logics but in the clarification of philosophical contents. Therefore, Weichselberger formulates the following confine: "Every intrinsic theory of the logical probability concept can be considered as an attempt to create a *formalized* and *quantitative theory of induction*." (Weichselberger [70, second page of Section 1.4] [translated by the authors; emphasis by Weichselberger])

4.4. When was Weichselberger's first encountering with imprecise probabilities?

Up to now, we did not succeed in exactly dating the begin of Weichselberger's reflection on imprecise probabilities. We think that it is not too speculative to relate it directly to the failure of precise methods in handling logical probability, in particular expressing the probabilistic inversion described above. Apparently, in that time, Weichselberger did not make any attempt to use set-valued probabilities also in the context of one-place probability, where events are evaluated.

Weichselberger's published work in the late fifties and sixties is positioned exclusively in the context of social and official statistics (Weichselberger [48,54,55]) and in traditional statistical methodology [46,47,23,53], including time series analysis (Weichselberger [49,50]). From today's perspective, the title "On parameter estimation in contingency tables with given marginal sums" [translated by the authors] of the two parts of the paper Weichselberger [46] sounds like an early application of the idea of natural extension, producing imprecise probabilities from given marginals. However, in that work Weichselberger considers a different question, namely the maximum likelihood estimation of inner cell probabilities of a contingency table from a sample, given the marginal frequencies.[11]

5. Probability intervals, *uncertainty in knowledge-based systems*

5.1. Uncertainty calculi for experts systems and their rejection

In the mid eighties of the last century, Weichselberger's research experienced a shift, which gave his interests in imprecise probabilities new impetus. He had been engaged in-depth in the vivid discussion about modelling uncertain expert knowledge in artificial intelligence, more specifically the treatment of uncertainty in what Weichselberger & Pöhlmann call a *diagnostic system*: An "expert system, which relies upon empirical interdependences for drawing its conclusions and consequently requires the treatment of uncertainty." [68, p. 1]

With many other researchers mainly from computer science, Weichselberger agreed that the problem how to model uncertain expert knowledge produces a big challenge, where the concept of probability, in its traditional form, reaches its limits. However, he also warned not to throw out the baby with the bath water and end up in arbitrary conclusions, by leaving the field to ad hoc calculi emerging at that time.

Mainly three alternatives to traditional probability theory had been discussed widely at that time (see also, e.g., Kruse et al. [28]):

- the *certainty factors* as developed in the expert system MYCIN, which evolved within the Stanford Heuristic Programming Project, documented in Buchanan and Shortliffe [9], see also Shortliffe and Buchanan [36],
- the Dempster–Shafer theory of *belief functions* [35], which has been developed further in the area of modelling uncertain knowledge, without reliance on its probabilistic and statistical roots (e.g., Dempster [15]),[12]
- and the theory of *fuzzy sets* introduced by Zadeh [71], which allows a gradual membership of an object in a set, associating with every such object a membership value between 0 and 1.[13]

In their book, Weichselberger and Pöhlmann [68] reject all three methods.

Fuzzy sets were left out of consideration for pragmatic reasons, based on a argument of scientific parsimony: "We believe that the use of interval estimates produces a degree of freedom large enough to distinguish between situations which may be relevant for the use in diagnostic systems. The combination of the theory of fuzzy sets with the methods proposed here would inevitably lead to further complications of these methods and consequently result in an impediment to their application." [68, p. 4]

The other two approaches, the Dempster–Shafer rule and the MYCIN certainty factors, were studied in some detail in Section 3.4 and 3.5 of the book, respectively, and refuted as potentially producing misleading conclusions. Combinations rules of the Dempster–Shafer type lacked a probabilistic justification and "[...] do not use all the information which may be relevant in order to draw conclusions from different pieces of evidence" [68, p. 54]. Therefore they "[...] must inevitably produce counter-intuitive effects in some situations [...]" (ibid.). The method of certainty factors, "[...] as an attempt to

[11] The setting is motivated by questions from official statistics when the marginal frequency distributions of two variables are known from a complete census, but in addition are to be supplemented by estimates of the joint distribution obtained from a sample, such that the estimators take the knowledge on the marginal distributions fully into account. The problem is considered in some extended form in Weichselberger [55].

[12] See also Denoeux [16] for a collection of some important work on the Dempster–Shafer theory published in the International Journal of Approximate Reasoning.

[13] For a detailed history of the theory of fuzzy sets see Seising [34]. An example of a successful application of fuzzy sets is the medical diagnostic system CADIAG-II [34, Chapter 7].

develop a special language for the communication between experts and an expert system [...]" is judged to remain obscure and thus to be "[...] a failure as it does not meet the basic requirement of introducing a new language: an exact description of the concepts used." [68, p. 64f.]

Already in their first chapter, Weichselberger and Pöhlmann argue:

First of all it must be stated that although the basic ideas prevailing in some considerations about diagnostic systems sound convincing, they violate fundamental requirements for reasonable handling of uncertainty. [... We] shall demonstrate that negligence with respect to [... some basic principles] may result in the inclusion of information into a diagnostic system which is equivalent to ruining it. [68, pp. 1–2]

Because these alternative methods to handle uncertainty in diagnostic systems failed, Weichselberger has understood it as a fundamental question for the discipline of statistics whether statistics can contribute here. Weichselberger stood for a very clear position: there will be an important contribution of statistics and probability in this area, if, but also radically only if, the concept of probability is ready to overcome the dogma of precision.

5.2. Reconciling probability theory and uncertainty calculi

Against the background, the book *A Methodology for Uncertainty in Knowledge-Based Systems* [68], published by Weichselberger together with his post-doctoral researcher Sigrid Pöhlmann, aims at reconciling probability theory with the objectives of flexible modern uncertainty calculi. There, Weichselberger and Pöhlmann develop, in the context of a prototypical special case,[14] a neat probabilistic alternative to handle different sources of information in diagnostic systems. It has to be consistently based on a generalisation of probability, synthesising the well-founded concept of probability with the flexibility needed for modelling uncertain knowledge:

[...An] argument against a possible application of probability theory [, understood in its traditional, precise form here,] in diagnostic systems is as follows: While probability theory affords statements, using real numbers as measures of uncertainty, the informative background of diagnostic systems is often not strong enough to justify statements of this type. This is indeed a true concern of the conception of diagnostic systems not met by probability theory in its traditional form. However, it is possible to expand the framework of probability theory in order to meet these requirements without violating its fundamental assumptions. In our study we shall present elements of a systematic treatment of problems of this kind and refer to related theories. Therefore we believe that the weakness of estimates for measures of uncertainty as used in diagnostic systems represents a stimulus to enrich probability theory and the methodological apparatus derived from it, rather than an excuse for avoiding its theoretical claims. [68, p. 2]

Technically, Weichselberger & Pöhlmann do not yet use interval probability in full generality, but confine themselves to the case which they call *PRI* from *PRrobability Intervals*.[15] There, an interval-valued probability is assigned to the singletons only, and natural extension is applied to calculate the probabilities of the other events. Moreover, speaking often of "interval *estimates* of probabilities" [emphasis by the authors], Weichselberger & Pöhlmann implicitly rely exclusively on the sensitivity analysis (epistemic) point of view of imprecise probabilities.

The book was published one year before Peter Walley's book [40] on general imprecise probability appeared. In Weichselberger and Pöhlmann's book the notions of R- and F-probability ("R" for *reasonable*, corresponding to *avoiding sure loss* to use Walley's terminology, and "F" for *feasible*, corresponding to *coherent*) were developed for the first time. Having been perceived mainly in the uncertainty modelling community (see Section 8), the book was also criticized strongly as "a little too unfinished" and too example-based in a review in the Journal of the American Statistical Association [44]. Convenient expressions to work with PRIs were extended in Weichselberger [60, Chapter 3.3 and Appendix A.5]. The construction of least favourable pairs for testing hypotheses described by PRIs is considered in Martin Gümbel's dissertation [20], supervised by Weichselberger.

6. Interval probability: *Elementare Grundbegriffe ...*

6.1. The book and the ISIPTA '99 paper including its IJAR extension

Immediately after having finished the book with Pöhlmann, Weichselberger started to develop the theory of interval probability as a "one-place assignment", i.e. assigning probability to events, in its generality.[16] No later than 1992, a first version of a book was completed, already containing the core concepts of the theory. The material grew and grew in its dimensions, and Weichselberger decided to split the book project into three volumes.

[14] The general case was later solved in Pöhlmann's Habilitation thesis [29].

[15] See Campos, Huete, and Moral [14] for an independent development of almost the same framework.

[16] Weichselberger, however, always has been stressing the importance of the "two-place concept" (logical probability with premises and conclusions as functional arguments, see Section 4) as the ideal, calling it still "[...] without doubt the most challenging [...]" concept (Weichselberger [60, p. 33] [translated by the authors]). Unless mentioned differently, the term *probability* is used throughout this section in its one-place meaning as probability of events.

Finally, in 2001 the first volume, *Elementare Grundbegriffe einer allgemeineren Wahrscheinlichkeitsrechnung I: Intervallwahrscheinlichkeit als umfassendes Konzept* (Elementary Foundations of a More General Calculus of Probability I: Interval Probability as a Comprehensive Concept [translated by the authors]) [60], appeared.[17] Soon this book became Weichselberger's most influential publication, together with the paper *The theory of interval probability as a unifying concept for uncertainty* [59], which arose from his ISIPTA '99 contribution and serves as an English language reference summarizing some of the main concepts (see also Section 8).

The title of the book, an immediate allusion to Kolmogorov's *Grundbegriffe*... [25] founding traditional probability theory, formulated the research program. Weichselberger develops thoroughly the theory of interval probability by generalizing the Kolmogorovian concept to interval-valued assignments. The book consists of four main chapters.[18]

The first chapter elaborates the background of the theory. It starts with embedding the theory into the historical development of the concept of probability, including other generalizations of probability. Then motives for the paradigmatic shift from traditional probability to interval probability and major objectives of the theory are discussed in-depth.

The second chapter contains the axioms of R- and F-probability. Weichselberger characterises interval-valued assignments $P(\cdot) = [L(\cdot), U(\cdot)]$ on a σ-field $\mathcal{A}$ by their relation to the set $\mathcal{M}$ of classical probabilities (in the sense of Kolmogorov) $p(\cdot)$ they induce. If this set is not empty, then $P(\cdot)$ is an *R-probability*, and $\mathcal{M}$ is its *structure*. An R-probability is an *F-probability* if $P(\cdot)$ and the structure uniquely correspond to each other by

$$L(A) = \inf_{p(\cdot)\in\mathcal{M}} p(A) \quad \text{and} \quad U(A) = \sup_{p(\cdot)\in\mathcal{M}} p(A), \quad \forall A \in \mathcal{A}.$$

In the light of Walley's lower envelope theorems, R-probability and F-probability technically correspond, in essence, to lower and upper probabilities avoiding sure loss and being coherent, respectively [40, Chapters 2 and 3], where, however, Weichselberger in the spirit of Kolmogorov, demands σ-additivity.

In conformity with Walley, Weichselberger stresses that there is no need to require additional restrictive properties (like two- or total monotonicity of the lower bound), but in contrast to him, Weichselberger focuses on interval-valued assignments of *events*, instead of random variables and gambles, respectively. For Weichselberger, probability of events is the constitutive entity (of a one-place probability assignment[19]); he sees expectations and previsions, respectively, as derived entities, explicitly needing an underlying metrical scale.

The most important difference for Weichselberger to Walley is that his axiomatisation is, just as the Kolmogorovian approach in traditional probability theory, strictly independent of any interpretation of probability. By this, he emphasises, it provides a sound mathematical basis for expressing all different interpretations of (one-place) generalized probabilities, from subjective to frequentist, which eventually is the key to overcome the methodological antagonisms in statistical inference.

Chapter 2.6 reflects on decision criteria based on probabilistic evaluations of events. There Weichselberger also argues that behaviour following Hurwicz-like criteria (e.g. Huntley, Hable, Troffaes [24, p. 193]) challenges Walley's betting interpretation of imprecise probabilities, which he judges to rely solely on a Γ-maximin point of view.[20]

Chapter 3 generalizes the setting to situations where the limits of an interval probability firstly are only specified on certain subsets of the σ-field $\mathcal{A}$, and then natural extension is applied (*partially determinate probability*). This gives rise to a list of interesting special cases, including PRIs (see Section 5) and a kind of general p-boxes (*cumulative R-/F-probability*[21]). Supplementing natural extension, which already appears in Weichselberger and Pöhlmann [68] (*derived F-PRI*), Weichselberger also proposes a *cautious standpoint* to proceed from a given R-probability $[L(\cdot), U(\cdot)]$ that is not F-probability to a uniquely defined F-probability $[L^*(\cdot), U^*(\cdot)]$, now such that the original limits $L(\cdot)$ and $U(\cdot)$ are always respected, in the sense that $L^*(\cdot) \le L(\cdot)$ and $U^*(\cdot) \ge U(\cdot)$.

Specific issues of interval probabilities on finite spaces are in the focus of Chapter 4, see also Weichselberger [58]. In Chapter 4.1 algorithms are developed to check whether assignments constitute R- and F-probability, as well as to calculate the natural extension and its counterpart from the cautious standpoint. Interestingly, linear programming is here not only utilized powerfully for calculations, but also, by duality results, as a mathematical tool for elegant proofs.

6.2. *Preceding first contributions to general interval probability; strongly related work and co-operations*

In this section we collect some of Weichselberger's activities when working on his book.

The axioms and further core elements of his theory were presented at several workshops, including a workshop in June 1993 honouring Peter J. Huber [58], the Second Gauss Symposium in August 1993 [56], and a workshop on the foundations

[17] The book title has the addendum "unter Mitarbeit von [in cooperation with] T. Augustin und (and) A. Wallner", which tributes to the special way the book was written: Augustin entered the project in 1993, Wallner in 1995, both as young PhD students and assistants. They were intensively engaged with the book, but not as co-authors (Wallner, and to an even smaller extent Augustin, contributed only some very short and clearly marked parts of the book, listed in Weichselberger [60, p. x]), but as critical discussions partners. Weichselberger extended and developed further the theory step by step, and in several meetings per week these steps were immediately and intensively discussed.

[18] See also the review by Coolen [13].

[19] See also Footnote 16.

[20] See also Coolen [13, p. 254].

[21] Compare also Destercke, Dubois, and Chojnacki [17] for a related concept.

Fig. 2. Participants of the Foundations of Statistics Workshop organized by Frank Hampel in 1994: From left to right: Walley, Goldstein, Smets, Coolen, Weichselberger, Morgenthaler, Hampel, Augustin (photo kindly provided by Frank Coolen).

of statistics in September 1994 in Zurich organized by Frank Hampel. By that workshop and an associated research retreat to the mountains nearby, Hampel connected researchers interested in the foundations of statistics (see also Fig. 2), who only partially knew each other personally. The participants' excited discussions had a sustainable impact on their further research. The relationship between Weichselberger (and Augustin) and Frank Coolen remained particularly close thereafter.

In 1995 also a paper on the implications of the rich framework of interval probability on sampling appeared (Weichselberger [57], see also Weichselberger [60, Chapter 4.3]), which in our eyes by far did not receive the attention it actually deserves. Only with interval probability it becomes possible to express the distinction between different types of symmetry, called *epistemic* versus *physical symmetry* by Weichselberger. Epistemic symmetry relies merely on the lack of knowledge of asymmetry, and therefore justifies a novel type of uniform probability that is based on interval-valued assignments. In contrast, for physical symmetry knowledge has to be available actively supporting symmetry. Only the latter in its purest form justifies the use of precise, traditional probabilities. By these concepts, Weichselberger develops nothing less than a generalization of the principle of insufficient reason, replacing precise uniform probability by a continuum of uniform probabilities, adequately expressing the knowledge on the system under consideration.

In 1996 Weichselberger [58] presents important aspects of his linear programming based techniques to handle computationally, but also to characterise interval probability, which build the basis for a larger part of Weichselberger [60, Chapter 4] (see the previous subsection).

Decision theoretic implications of imprecise probabilities are discussed in 1998 in a contribution [66] to a Festschrift honouring Weichselberger's Munich long-standing colleague Hans Schneeweiß, working out how interval probability provides an immediate description of the preferences observed in Ellsberg's seminal experiments [18], violating the axioms of traditional subjective utility theory.

As a preparation for the third volume, which was originally devoted to statistical implications of interval probability, the Huber-Strassen theory on robust testing of hypotheses described by neighbourhood models had been intensively discussed by the members of Weichselberger's chair and Helmut Rieder, who spent one semester at LMU Munich in 1994. Augustin, who originally had started a dissertation about the historical roots of imprecise probability, took over the topic and developed under Weichselberger's supervision a Neyman–Pearson theory under general interval probability, where the hypotheses are described by F-probability instead of two-monotone capacities. In his thesis [1] it is shown that Weichselberger's condition of continuity of F-fields [60, p. 152f.] is both necessary and sufficient for the structure to be uniformly dominated. Furthermore, Augustin derives results on different types of least favourable pairs (published in a generalized form for the first ISIPTA and in Augustin [2], based on it) and a representation of the optimal test by a single linear program (published later in a decision theoretic context in Augustin [3,4] and in further extended form by Utkin and Augustin [37]), including a Neyman–Person lemma form obtained from duality arguments.

6.3. Further planned volumes, work on interval probability after the book

When the book appeared, a second volume was already in a pretty advanced stage. Originally it was devoted to a closer study of types of assignments that lead to two- or totally-monotone capacities (probability intervals, cumulative probabilities, belief-functions), concepts of conditional probabilities and independence, parametric statistical models and a law of large numbers.[22]

In Weichselberger's ISIPTA '01 contribution [61], *indicator fields* are studied, i.e. interval probabilities that can be understood as basic building blocks for more complex models. In 2002, Lev Utkin visited the Weichselberger chair for almost two years, and a very close relationship with Weichselberger (and Augustin) was established that has endured since then.

At ISIPTA '03 (cf. Weichselberger and Augustin [67]), Weichselberger presented his research on conditional probability. He strongly argued in favour of the idea that there cannot be exclusively a single concept of conditional probability. Several distinct concepts are needed, which happen to coincide in the case of a precise probability. In particular, he elaborates his – rather controversially perceived – *canonical concept of conditional interval probability*, derived from a canon of desirable properties, like a commutative combination of marginals and conditional probabilities.[23]

In autumn of that year, Weichselberger abruptly stopped his research on one-place probability and radically turned all his interest to the foundation of logical probability again (see Section 7).

Anton Wallner, who had worked together very closely with Weichselberger all that time (see also Footnote 17), first continued his research on the one-place interval probability and prepared a dissertation under Weichselberger's supervision [41]. There, he develops a series of characterisations of interval probabilities in general as well as of uniform interval probabilities, and studies neighbourhood models based on distorted probabilities. Furthermore he presents a proof that also under general interval probability the structure of an R-probability on a space with cardinality k has, interpreted as a polyhedron in $\mathbb{R}^{k-1}$, at most $k!$ vertices. Related articles, presented at ISIPTA '03 and '05, are Wallner [42] and Wallner [43].

7. Logical probability II

All the development of one-place interval probability described in the previous two sections, as interesting it may be on its own, has been understood by Weichselberger mainly as a preparation for his concept of logical probability, and thus for his general inference theory. Therefore, from 2003 on Weichselberger had devoted all his energy to this topic.

Supported by Wallner, Weichselberger started to (re)build a framework for logical probabilities, now finding a neat basis in the theory of interval probability, pushing the vision of a comprehensive theory of inference closer to reality. Many of the constituents already mentioned in his inaugural speech as rector in Berlin [51], see also Section 4 above, are revisited in the light of the new foundation. In a special session on the symmetric theory at ISIPTA '09 [64, p. 9], he roughly characterises his major objective in simplified terms as follows:

> *A comprehensive methodology of probabilistic modelling and statistical reasoning, which makes possible hierarchical modelling with information gained from empirical data.*
> *To achieve the goals of Bayesian approach – but without the pre-requisite of an assumed prior probability. [64, p. 3]*

First results had been presented to a wider audience at a workshop 2004 in Munich, on the occasion of Weichselberger's 75th birthday, and at ISIPTA '05 in Pittsburgh with a paper [62] and a tutorial. The 2007 ISIPTA [63] paper is devoted to an autonomous concept of independence, which turns out to be a crucial basis for the axiom system underlying logical probability. In 2009, Weichselberger presented his theory in July at the special session of ISIPTA '09 [64], mentioned above, and in September at a special session of the WPMSIIP workshop in Munich devoted to his 80th birthday. According to the best of our knowledge, these two presentation were the last public presentations he gave. The last typeset version of Weichselberger's manuscript on the symmetric theory, that is available to us, dates at August 12, 2009.

Although this work [65] of more than 250 pages remains incomplete in many details, the core of the theory is clearly visible: The fundamental idea of logical probability as a two-place function formalizing the reasoning from a premise to the conclusion is framed in a system of axioms (Weichselberger [65, Chapter 4], see also Weichselberger [64, p. 8] for a short sketch), while the inference is developed in the context of a duality theory (Weichselberger [65, Chapter 6], for some aspects see also Weichselberger [64]). Also the idea of a frequency interpretation of logical probability could be made rigorous (Weichselberger [65, Chapter 5], see also Weichselberger [64, p. 9]).

More than ever, Weichselberger stresses the symmetric character of his theory. Relying on a theory of logical probability that puts probability statements on both directions of reasoning on propositions that are "[...] conceptually at the identical level [...]" [65, p. 3] offers vast novel opportunities for a new theory of statistical inference:

> *It is to be expected that the practical advantages of the symmetric theory can be observed in the first line in complex multi-layer models, where the results of a statistical analysis are not just stated but are demonstrated in all their consequences – as parameters*

[22] Some concepts are already briefly sketched in Weichselberger [59].
[23] Some aspects are already discussed in Weichselberger [59, Section 3].

Table 1
Numbers of citations of two of Weichselberger's main works by subject category, data: Google Scholar 11.3.2018, subject categories classified by the authors.

	Weichselberger & Pöhlmann (1990)			Weichselberger (2000, IJAR)		
	Journal	Monograph	Collection	Journal	Monograph	Collection
Mathematics	0	1	0	14	1	0
Uncertain knowledge[26]	26	2	11	50	1	43
Engineering	9	1	1	52	10	31
Economic sciences	4	1	8	12	8	9
Statistics	2	2	3	20	5	6
Philosophy	0	0	0	5	0	0

of further probabilistic models. Compared to the handling of such problems in Bayesian approaches, the fundamental difference is that one does not have to select a prior; this guarantees the universal validity of statements. The benefit compared to the classical theory of objective orientation is anyway that models of an hierarchical type are possible at all. (Weichselberger [65, p. 4]) [translated by the authors])

8. A first bibliometric analysis

To obtain some impression of the reception of Weichselberger 's work, we conducted a first citation analysis of some major work of all periods, namely the book Weichselberger and Pöhlmann [68], the paper Weichselberger [59] related to his 2001 book [60], and the work on logical probability. For this small study we preferred Google Scholar^{TM} over other data bases, in order to include a wider range of proceedings and at least some books. We distinguished three types of publication media: journal, proceedings and other collections, and monographs, while other media like technical reports, patents or presentation slides are not considered. We classified the publications according to the orientation of the journal, collection or the book into the following categories:

1. *mathematics*[24]
2. modelling *uncertain knowledge*, artificial intelligence, machine learning, system theory, computer science, cybernetics, including all work published in the International Journal of Approximate Reasoning and the ISIPTA-Proceedings,
3. *engineering* including reliability and safety analysis, science
4. *economic sciences*, including decision analysis, operations research, business administration/management science, economics, finance
5. *statistics* (in a narrower sense)
6. *philosophy*.

Before we present and briefly discuss our results, we have to put them into the correct place, asking to treat them with some caution. First, quite evidently, the coverage of the search can not be complete. Secondly, and probably even more important, our categorization scheme is inevitably vague and thus suffers from some arbitrariness. The category statistics was understood in a narrower sense, and did not include work from reliability analysis or from machine learning or modelling uncertain knowledge, although of course the transitions between statistics and these research areas are quite smooth.[25] Nevertheless some tendencies should be visible.

Clearly, the reception of both major works is rather different. As Table 1 shows, the latter work is much more often cited. Moreover, while Weichselberger and Pöhlmann [68] has only some pronounced influence in uncertainty modelling, Weichselberger [59] is also recognized strongly in engineering, has a much larger impact in statistics, and received some attention in mathematics and philosophy.

Up to now, Weichselberger's work on logical probability (see Sections 4 and 7), however, has been cited quite rarely, and its active reception has not yet left the circle of his close collaborators.

9. Concluding remarks

We presented a preliminary summary of Kurt Weichselberger's contribution to the theory of imprecise probability and statistical inference. As already emphasized, this paper is a report on current research within the HiStaLMU project, an interdisciplinary project involving statisticians and historians of science to chronicle the history of statistics at LMU Munich. Concerning the research on Weichselberger's scientific biography, the next practical step is to build up the necessary infrastructure by establishing an archive of his office estate and the material kindly provided by his family.

[24] The text in italics is used to describe the rows in Table 1.

[25] This vagueness of categorization is also corroborated by the fact that quite a number of authors cross borders of disciplines, most notably between statistics and modelling uncertain knowledge, mathematics and engineering.

[26] These numbers including 3 papers in IJAR and 1 paper in ISIPTA proceedings citing Weichselberger and Pöhlmann [68] and 14 papers in IJAR and 16 papers in ISIPTA proceedings citing Weichselberger [59, IJAR].

Far beyond the historical interest, a detailed rework of Weichselberger's unfinished opus and his scattered results will enable a deeper discussion of his scientific inheritance. His results and ideas provide a big challenge, still promising a substantial impact on, nay, a paradigmatic shift of, probability and statistics.

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References[27]

[1] T. Augustin, Optimale Tests Bei Intervallwahrscheinlichkeit (Optimal Tests under Interval Probability [oET]), Vandenhoeck and Ruprecht, Göttingen, 1998.
[2] T. Augustin, Neyman–Pearson testing under interval probability by globally least favorable pairs: reviewing Huber–Strassen theory and extending it to general interval probability, J. Stat. Plan. Inference 105 (2002) 149–173. Based on an ISIPTA '99 paper.
[3] T. Augustin, Expected utility within a generalized concept of probability: a comprehensive framework for decision making under ambiguity, Stat. Pap. 43 (2002) 5–22.
[4] T. Augustin, Optimal decisions under complex uncertainty: basic notions and a general algorithm for data-based decision making with partial prior knowledge described by interval probability, Z. Angew. Math. Mech. 84 (2004) 678–687.
[5] T. Augustin, Kurt Weichselberger (13.4.1929–7.2.2016) (Obituary on Kurt Weichselberger for SIPTA), see: https://statsoz-neu.userweb.mwn.de/research/MemoryKurtWeichselberger/Weichselberger.pdf, 2016 [last access: April 4, 2018].
[6] T. Augustin, R. Seising, Kurt Weichselberger's contribution to imprecise probabilities, in: A. Antonucci, G. Corani, I. Couso, S. Destercke (Eds.), Proceedings of the Tenth International Symposium on Imprecise Probability: Theories and Applications, ISIPTA '17, Lugano, in: Proceedings of Machine Learning Research, vol. 62, PMLR, 2017, pp. 13–24, http://proceedings.mlr.press/v62/augustin17a.html.
[7] T. Augustin, T. Parstorfer, N. Krem, G. Mechteridis, R. Seising, Interviews on the History of the Department of Statistics at LMU Munich: Christa Jürgensonn, Hans Schneeweiß and Christina Schneider, HiStaLMU Project, LMU Munich, 2013/2014.
[8] G. Boole, An Investigation of the Laws of Thought on Which Are Founded the Mathematical Theories of Logic and Probabilities, Walton and Maberly, London, 1854.
[9] B. Buchanan, E. Shortliffe (Eds.), Rule Based Expert Systems: The MYCIN Experiments of the Stanford Heuristic Programming Project, Addison-Wesley, Reading, MA, 1984.
[10] R. Carnap, Logical Foundations of Probability, University of Chicago Press, Chicago, 1950.
[11] R. Carnap, The Nature and Application of Inductive Logic, University of Chicago Press, Chicago, 1951.
[12] R. Carnap, W. Stegmüller, Induktive Logik und Wahrscheinlichkeit, Springer, Wien, 1959.
[13] F. Coolen, Book review "Elementare Grundbegriffe einer Allgemeineren Wahrscheinlichkeitsrechnung, vol. I, Intervallwahrscheinlichkeit als umfassendes Konzept" by Weichselberger, J. R. Stat. Soc., Ser. D 52 (2003) 253–254.
[14] L. de Campos, J. Huete, S. Moral, Probability intervals: a tool for uncertain reasoning, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 2 (1994) 167–196.
[15] A. Dempster, Upper and lower probabilities induced by a multivalued mapping, Ann. Math. Stat. 38 (1967) 325–339.
[16] T. Denoeux, 40 years of Dempster–Shafer theory, Int. J. Approx. Reason. 79 (2016) 1–6.
[17] S. Destercke, D. Dubois, E. Chojnacki, Unifying practical uncertainty representations – I: generalized p-boxes, Int. J. Approx. Reason. 49 (2008) 649–663.
[18] D. Ellsberg, Risk, ambiguity, and the Savage axioms, Q. J. Econ. 75 (1961) 643–669.
[19] H. Fischer, R. Seising (Eds.), Wahrscheinlichkeit und Statistik: Geschichte, Kulturen, Menschen. [Probability and Statistics: History, Cultures, People], Springer, Heidelberg, in preparation.
[20] M. Gümbel, Über die Effiziente Anwendung Von F-PRI: Ein Beitrag zur Statistik Im Rahmen Eines Allgemeineren Wahrscheinlichkeitsbegriffs (On the Efficient Application of F-PRIs: A Contribution to Statistics in the Framework of a More General Concept of Probability), Pinus, Augsburg, 2009.
[21] T. Hailperin, Boole's Logic and Probability: A Critical Exposition from the Standpoint of Contemporary Algebra, Logic and Probability Theory, second edition, North-Holland, Amsterdam, 1986.
[22] W.K. Härdle, A.B. Vogt, Ladislaus von Bortkiewicz: statistician, economist and a European intellectual, Int. Stat. Rev. 83 (2015) 17–35.
[23] S. Heiler, K. Weichselberger, Über den Permutationstest und ein daraus ableitbares Konfidenzintervall (On the permutation test and a confidence interval derived from it), Metrika 14 (1969) 232–248.
[24] N. Huntley, R. Hable, M. Troffaes, Decision making, in: T. Augustin, F. Coolen, G. de Cooman, M. Troffaes (Eds.), Introduction to Imprecise Probabilities, Wiley, Chichester, 2014, pp. 190–206.
[25] A. Kolmogorov, Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933, English translation: Foundations of Probability, Chelsea, Providence, RI, 1950.
[26] W. Krämer, Interview mit [with] Heinz Grohmann, AStA Wirtsch. Sozialstat. Arch. 8 (2014) 269–277.
[27] N. Krem, Die Geschichte des Instituts für Statistik in München von 1971 bis 2002 (The History of the Department of Statistics in Munich from 1971 to 2002), Master Thesis: Department of Statistics, LMU Munich, 2014.
[28] R. Kruse, E. Schwecke, J. Heinsohn, Uncertainty and Vagueness in Knowledge Based Systems: Numerical Methods, Springer, Berlin, 1991.

[27] In the references all German titles are also given in English. "oEt" stands for "original English title" in cases where one is given; all other titles are translated by the authors.

[29] S. Pöhlmann, Kombination von unsicherem Wissen in Form von Wahrscheinlichkeitsintervallen (Combination of Uncertain Knowledge Given in Form of Probability Intervals), Habilitationsschrift [Habilitation Thesis], LMU Munich, 1995.
[30] H. Rinne, B. Rüger, H. Strecker (Eds.), Grundlagen der Statistik und ihre Anwendungen (Foundations of Statistics and its Applications), Festschrift für [for] Kurt Weichselberger, Physika, Heidelberg, 1995.
[31] B. Rüger, Kurt Weichselberger, in: H. Rinne, B. Rüger, H. Strecker (Eds.), Grundlagen der Statistik und ihre Anwendungen (Foundations of Statistics and its Applications), in: Festschrift für [for] Kurt Weichselberger, Physika, Heidelberg, 1995, pp. 3–14.
[32] B. Rüger, Nachruf auf Kurt Weichselberger (Obituary on Kurt Weichselberger), see: https://statsoz-neu.userweb.mwn.de/research/MemoryKurtWeichselberger/Weichselberger.pdf, 2016 [last access: April 4, 2018].
[33] L. Savage, The Foundations of Statistics, Wiley, New York, 1954.
[34] R. Seising, The Fuzzification of Systems. The Genesis of Fuzzy Set Theory and Its Initial Applications Developments up to the 1970s, Springer, Berlin, 2007.
[35] G. Shafer, A Mathematical Theory of Evidence, Princeton University Press, 1976.
[36] E. Shortliffe, B. Buchanan, A model of inexact reasoning in medicine, Math. Biosci. 23 (1975) 351–379.
[37] L. Utkin, T. Augustin, Powerful algorithms for decision making under partial prior information and general ambiguity attitudes, in: F. Cozman, R. Nau, T. Seidenfeld (Eds.), Proceedings of the Fourth International Symposium on Imprecise Probability: Theories and Applications, ISIPTA '05, Pittsburgh, Carnegie Mellon University, Pittsburgh, 2005, pp. 349–358.
[38] A. Vogt, Statistik zwischen Mathematik und Ökonomie: institutionalisierte Statistik als Teil der "Nationalökonomie" und Versicherungsmathematik bis 1960 im nationalen und internationalen Kontext [Statistics between mathematics and economics: institutionalized statistics as a part of national economics und insurance mathematics until 1960 in national and international contexts]. In H. Fischer and R. Seising, editors, Wahrscheinlichkeit und Statistik: Geschichte, Kulturen, Menschen. [Probability and Statistics: History, Cultures, People]. Springer, Heidelberg, in preparation.
[39] P. von der Lippe, S. Schmerbach, Mehr wirtschaftsstatistik in der statistikausbildung für volks- und betriebswirte (More economic statistics in the statistical education for students of economics and management science), Allg. Stat. Arch. 87 (2003) 335–344.
[40] P. Walley, Statistical Reasoning with Imprecise Probabilities, Chapman & Hall, London, 1991.
[41] A. Wallner, Beiträge zur Theorie der Intervallwahrscheinlichkeit: der Blick über Kolmogorov und Choquet hinaus (Contributions to the Theory of Interval Probability: The View beyond Kolomogorv and Choquet), Kovač, Hamburg, 2002.
[42] A. Wallner, Bi-elastic neighbourhood models, in: J. Bernard, T. Seidenfeld, M. Zaffalon (Eds.), Proceedings of the Third International Symposium on Imprecise Probability: Theories and Applications, ISIPTA '03, Lugano, Carleton Scientific, Waterloo, 2003, pp. 593–607.
[43] A. Wallner, Extreme points of coherent probabilities on finite spaces, Int. J. Approx. Reason. 44 (2007) 339–357. Based on an ISIPTA '05 paper.
[44] L. Wasserman, Book review: "A methodology for uncertainty in knowledge-based systems" by Weichselberger & Pöhlmann, J. Am. Stat. Assoc. 86 (1991) 546–547.
[45] K. Weichselberger, Bernstein-Polynomapproximation in Höheren Räumen (Bernstein Polynomial Approximation in Higher Spaces), Dissertation, Vienna University, 1953.
[46] K. Weichselberger, Über die Parameterschätzung bei Kontingenztafeln, deren Randsummen vorgegeben sind (On parameter estimation in contingency tabels with given marginal sums) (part I and part II) Metrika 2 (1959) 100–130 and 198–229.
[47] K. Weichselberger, Über ein graphisches Verfahren zur Trennung von Mischverteilungen und zur Identifikation kupierter Normalverteilungen bei großem Stichprobenumfang (On a graphical method to separate mixture distributions and to identify truncated normal distributions under large sample size), Metrika 4 (1961) 178–229.
[48] K. Weichselberger, Kontrollen der Ergebnisse von Volkszählungen (Controlling the Results of Population Censuses), Habilitationsschrift [Habilitation Thesis], University of Cologne, 1962.
[49] K. Weichselberger, Über eine Theorie der gleitenden Durchschnitte und verschiedene Anwendungen dieser Theorie (On a theory of moving averages and various applications of this theory), Metrika 8 (1964) 185–230.
[50] K. Weichselberger, Analyse von Zeitreihen und Unregelmäßigkeiten (Analysis of time series and their irregularities), in: H. Strecker, W. Bihn (Eds.), Die Statistik in der Wirtschaftsforschung: Festgabe für Rolf Wagenführ (Statistics in Economics Research: Festschrift for Rolf Wagenführ), Duncker & Humblot, Berlin, 1967, pp. 435–475.
[51] K. Weichselberger, Einige Grundprobleme der Statistik und der Wahrscheinlichkeitstheorie: Rektoratsübergabe, 24.11.1967 (Some fundamental problems of statistics and probability: Rector's Inaugural Speech, Nov. 24), Technische Universität Berlin: Akademische Reden (Technical University of Berlin: Academic Speeches) 47 (1967) 34–50, Digitalized by Historische Kommission bei der Bayerischen Akademie der Wissenschaften see: www.historische-kommission-muenchen-editionen.de/rektoratsreden, 1968 [last access: last access April 4, 2018].
[52] K. Weichselberger, Gegenwart und Zukunft der Universität (Presence and future of the university), in: J. Broermann, P. Herder-Dorneich, F. Hermens, F. Mann, W. Schreiber (Eds.), Soziale Verantwortung. Festschrift für Goetz Briefs zum 80. Geburtstag (Social Responsibility. Festschrift for Goetz Briefs on the Occasion of his 80th Birthday), Duncker & Humblot, Berlin, 1968, pp. 645–663.
[53] K. Weichselberger, Einflußgrößenrechnung und ihre praktische Anwendung (Regression analysis and its practical application [oET]), Allg. Stat. Arch. 53 (1969) 35–50.
[54] K. Weichselberger, Genauigkeitsansprüche, Fehler und Kontrollen bei Volkszählungen (Precision requirements, errors and checks in population censuses [oET]), Allg. Stat. Arch. 54 (1970) 1–28.
[55] K. Weichselberger, Über die statistischen Eigenschaften von Korrekturen aufgrund repräsentativer Verfahrenskontrollen (On the statistical properties of corrections based on respresentative controls of methods), Metrika 17 (1971) 159–188.
[56] K. Weichselberger, Axiomatic foundations of the theory of interval probability, in: V. Mammitzsch, H. Schneeweiß (Eds.), Symposia Gaussiana; Proceedings of the 2nd Gauss Symposium, B: Statistical Sciences, de Gruyter, Berlin, 1995, pp. 47–64.
[57] K. Weichselberger, Stichproben und Intervallwahrscheinlichkeit (Sampling and interval probability), Ifo-Stud. 41 (1995) 653–676.
[58] K. Weichselberger, Interval probability on finite sample spaces, in: H. Rieder (Ed.), Robust Statistics, Data Analysis, and Computer Intensive Methods: in Honor of Peter Huber's 60th Birthday, Springer, New York, 1996, pp. 391–409.
[59] K. Weichselberger, The theory of interval probability as a unifying concept for uncertainty, Int. J. Approx. Reason. 24 (2000) 149–170. Based on an ISIPTA '99 paper.
[60] K. Weichselberger, Elementare Grundbegriffe einer allgemeineren Wahrscheinlichkeitsrechnung I: Intervallwahrscheinlichkeit als umfassendes Konzept (Elementary Foundations of a More General Calculus of Probability I: Interval Probability as a Comprehensive Concept), Physica, Heidelberg, 2001 (Unter Mitarbeit von Thomas Augustin und Anton Wallner [In cooperation with Thomas Augustin and Anton Wallner]).
[61] K. Weichselberger, The status of F-indicator-fields within the theory of interval-probability, in: G. de Cooman, T. Fine, T. Seidenfeld (Eds.), Proceedings of the Second International Symposium on Imprecise Probability: Theories and Applications, ISIPTA '01, Ithaca, Shaker, Maastricht, 2001, pp. 362–369.
[62] K. Weichselberger, The logical concept of probability and statistical inference, in: F. Cozman, R. Nau, T. Seidenfeld (Eds.), Proceedings of the Fourth International Symposium on Imprecise Probability: Theories and Applications, ISIPTA '05, Pittsburgh, SIPTA, Manno, 2005, pp. 396–405 (In cooperation with Anton Wallner).
[63] K. Weichselberger, The logical concept of probability: foundation and interpretation, in: G. de Cooman, J. Vejnarová, M. Zaffalon (Eds.), Proceedings of the Fifth International Symposium on Imprecise Probability: Theories and Applications, ISIPTA '07, SIPTA, 2007, pp. 455–463 (In cooperation with Anton Wallner).

[64] K. Weichselberger, Symmetric probability theory, presentation at a *Special Session at the Sixth International Symposium on Imprecise Probability: Theories and Applications (ISIPTA '09, Durham (UK))*, 2009.

[65] K. Weichselberger, Elementare Grundbegriffe einer allgemeineren Wahrscheinlichkeitsrechnung II. Symmetrische Wahrscheinlichkeitstheorie [Elementary Foundations of a More General Calculus of Probability II: The Symmetric Theory of Probability]. (Unpublished manuscript: last typeset version: version 49, August 12, 2009; 268 pages), 2016 (in cooperation with Anton Wallner).

[66] K. Weichselberger, T. Augustin, Analysing Ellsberg's paradox by means of interval-probabilty, in: R. Galata, H. Küchenhoff (Eds.), Econometrics in Theory and Practice (Festschrift for Hans Schneeweiß), Physica, Heidelberg, 1998, pp. 291–304.

[67] K. Weichselberger, T. Augustin, On the symbiosis of two concepts of conditional interval probability, in: J. Bernard, T. Seidenfeld, M. Zaffalon (Eds.), Proceedings of the Third International Symposium on Imprecise Probability: Theories and Applications, ISIPTA '03, Lugano, Carleton Scientific, Waterloo, 2003, pp. 608–630.

[68] K. Weichselberger, S. Pöhlmann, A Methodology for Uncertainty in Knowledge-Based Systems, Springer, Heidelberg, 1990.

[69] K. Weichselberger, A.-R. Wulsten, Preisindices für nicht-kommerzielle Forschung in der Bundesrepublik Deutschland 1968–1977 (Price indices for non-commercial research in the Federal Republic of Germany 1968–1977), Seminar für Spezialgebiete der Statistik, Universität München, 1978.

[70] K. Weichselberger, Wahrscheinlichkeitstheorien im Vergleich (Theories of Probability in Comparison). Fragments of an uncompleted manuscript, ca. 200 typeset pages, undated.

[71] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965) 338–353.