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Bi-directional h-index: A new measure of node centrality in weighted and directed networks

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ABSTRACT

This paper builds an index family, named bi-directional h-index, to measure node centrality in weighted directed networks. Bi-directional h-index takes the directed degree centrality as the initial value and iteratively uses more network information to update the node's importance. We prove the convergence of the iterative process after finite iterations and introduce an asynchronous updating process that provides a decentralized, local method to calculate the bi-directional h-index in large-scale networks and dynamic networks. The theoretical analysis manifests that the bi-directional h-index is feasible and significant for establishing a greater conceptual framework that includes some existing index concepts, such as lobby index, node's h-index, c-index and iterative c-index. An example using journal citation networks indicates that the bi-directional h-index is different from directed degree centrality, directed node strength, directed h-degree and the HITS algorithm in ranking node importance. It is irreplaceable and can reflect these measures of node's importance.

1. Introduction

Network analysis methods provide a valuable opportunity to improve information science research by exploring surface information to extract the deep structure and relationships (Newman, 2010). Real networks exhibit a heterogeneous nature with nodes playing far different roles in structure and function (Barabási, 2009; Barabási & Albert, 1999; Lü, Chen et al., 2016). A challenge for network analysis is the identification of key nodes within a network (Lü, Zhou, Zhang, & Stanley, 2016). As a matter of fact, identifying the important nodes helps to optimize the use of limited resources to facilitate information propagation (Morone & Makse, 2015), work out a fixed-point advertisement strategy based on influential spreaders for viral marketing (Akritidis, Katsaros, & Bozanis, 2011; Leskovec, Adamic, & Huberman, 2007), facilitate the access and spread of information in online social networks (Morone & Makse, 2015; Rabade, Mishra, & Sharma, 2014) and monitor exceptional events like mass mobilizations (González-Bailón, Borge-Holthoefer, Rivero, & Moreno, 2011).

The importance of a node is largely affected and reflected by the topological structure of the network it belongs to (Lü, Chen et al., 2016). The concept of centrality was proposed to characterize a node's importance according to the structure (Bonacich, 1987; Borgatti, 2005; Borgatti & Everett, 2006). This classic method of identification of a node's importance allows for wide applications independent of the specific dynamic processes under consideration. Recently, applying the h-index (Hirsch, 2005) in network science has aroused the attention of scholars, and some centralities based on the h-index with a good

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performance are proposed, e.g., node's h-index (Lü, Chen et al., 2016; Lü, Zhou et al., 2016), lobby index (Korn, Schubert, & Telcs, 2009), h-degree (Zhao, Rousseau, & Ye, 2011; Zhao & Ye, 2012), c-index and iterative c-index (Yan, Zhai, & Fan, 2013), and communication centrality (Zhai, Yan, & Zhang, 2013). These h-type methods are mostly defined in undirected or unweighted networks, but little research has been done in weighted directed networks. However, many typical information networks are weighted and/or directed networks, such as co-word networks, citation networks, collaboration networks, knowledge flow networks, hyperlink networks, question and answer (Q&A) networks and message forwarding networks in online social sites.

In weighted directed networks, the heterogeneity of links is not only reflected by the disparity of the strength, but also registered as the difference of the directions. Moreover, in directed networks, the directions of links usually characterize two kinds of reverse relations, e.g., cited or citing, and information acquisition or information diffusion (Zhao & Ye, 2012). So nodes in the directed network boast two sorts of importance according to the directions of links. There is the mutually reinforcing relationship (Kleinberg, 1999) between two kinds of reverse relations. However, the h-type methods proposed in directed networks cannot describe it. Thus, it might be interesting to explore a centrality measure based on the h-index in weighted directed networks, which can exhibit the mutually reinforcing relationship of two kinds of reverse relations.

In this paper, we discuss the extending of the h-index concept to quantify a node's importance in weighted directed networks, and then propose the concept of bi-directional h-index. The bi-directional h-index is similar to the HITS algorithm (Kleinberg, 1999) for dealing with the problem of the mutually reinforcing relationship of two kinds of reverse relations. It will retain the advantage that h-type centralities are easier to be calculated. To more precisely depict the importance of the neighboring nodes, we designed an iterative updating process to update the node's importance by means of the iterative refinement (Lü, Chen et al., 2016) or mutual enhancement effect (Wittenbaum, Hubbell, & Zuckerman, 1999). We proved the convergence of the iterative process within the finite iterations and call the iterative process "synchronous updating process." We further show that the convergence can be guaranteed even under an asynchronous updating process, thus allowing a distributed computing algorithm to deal with large-scale networks or dynamic networks. The bi-directional h-index will establish a greater conceptual framework of h-type centralities, for which the lobby index, node's h-index, c-index and iterative c-index can all be regarded as its special case.

This paper is organized as follows. Some related works are summarized in Section 2. We discuss theoretical aspects in Section 3. Specifically, we put forward the definition and properties of the bi-directional h-index in weighted directed networks, give the synchronous and asynchronous updating processes of the bi-directional h-index, and prove that they have the same limits. In Section 4, we apply the bi-directional h-index on a journal citation network and analyze its effects. Section 5 is the conclusion and discussion. The proofs of theorems and properties are provided in the Appendix A.

2. Related works

2.1. Weighted directed networks

Network science systematically analyzes individuals and their relationship as a whole in this complex phenomenon, with the individuals defined as the nodes and their relationships as the links in networks. Based on the heterogeneity of links, weights and directions, networks can be divided into four types, unweighted undirected networks, weighted networks, directed networks and weighted directed networks (Zhao & Ye, 2012). Many traditional centralities in network analysis, such as degree centrality (Freeman, 1979; Nieminen, 1974; Shaw, 1954), closeness centrality (Bavelas, 1950; Freeman, 1979; Sabidussi, 1966), betweenness centrality (Freeman, 1979) and eigenvector centrality (Bonacich, 1972), were initially designed for unweighted undirected networks.

The unweighted undirected networks are also called simple networks (Lü, Chen et al., 2016), which are simplified networks that ignore the difference of strength and direction of the links between the individuals and only consider whether the links exist. In many real systems, the interactions between nodes are usually not merely binary entities (either present or not). As a pivotal type of network, the weighted directed network distinguishes the strength and direction of the links between individuals, and thus can more accurately describe the individuals and their relationships (Opsahl, Agneessens, & Skvoretz, 2010). For easy distinction, the links are often called arcs in directed networks. Denote a weighted directed network by $G(V, E)$, where V and E are the sets of nodes and arcs, respectively. The weighted adjacency matrix is W, where $W = \{s_{ij}\}$ with $s_{ij} > 0$ if there is an arc from node i that points to node j with arc weight s_{ij}, otherwise $s_{ij} = 0$. Here the weight associated with a link quantifies the strength of the directed relationship from one node to another node. In fact, the first three categories of networks can also be regarded as special cases of the weighted directed networks. In unweighted networks, the weights of all the links are the same, and usually the weights are set as 1. In undirected networks, all the links are regarded as bi-directional arcs. As a result, developing more appropriate measures becomes a vital and interesting task in weighted directed networks.

2.2. H-index and h-type centralities

Generally speaking, a centrality measure assigns a real value to each node in the network, where the values produced are expected to provide a ranking of nodes subject to their importance (Lü, Chen et al., 2016; Newman, 2010). Due to the wide meanings of importance from different aspects, many methods have been proposed.

Recently, scholars found that applying the idea of h-index can reveal some interesting features in network science, and many h-type centralities have been proposed based on h-index. The h-index is one of the basic indicators in bibliometrics (Egghe, 2010). “A scientist has index h if h of his or her N_p papers have at least h citations each and the other $N_p–h$ papers have $\leq h$ citations each” (Hirsch, 2005). H-index simply and effectively measures the key part of a set of articles in a relatively natural way and is now widely used as an indicator to evaluate the lifetime achievements of a scientist, institution or country (Zhao et al., 2011). For example, h-index is used by many academic databases, such as the Web of Science and Scopus.

For convenience, we label the h-index algorithm as an operator H, which acts on finite real numbers $(x_1, x_2, \cdots, x_n)$ and returns an integer y, where y is the maximum integer such that there exist at least y elements in $(x_1, x_2, \cdots, x_n)$, each of which is no less than y. Without loss of generality, let $x_1 \geq x_2 \geq \cdots \geq x_n$, then $y = H(x_1, x_2, \cdots, x_n) = \max\{y|x_y \geq y\} \geq 0$. Here if $1 > x_1 \geq x_2 \geq \cdots \geq x_n$ or $n = 0$, then we define $y = H(x_1, x_2, \cdots, x_n) = 0$.

Based on the fact that a node’s importance is highly correlated to its capacity to impact the behaviors of its surrounding neighbors, Korn et al. (2009) first analyzed networks using h-index and proposed the lobby index of nodes based on the degree centralities of neighbors in simple networks. The lobby index $l(i)$ of a node i is defined to be the maximum value k such that there exists at least k neighbors of degree no less than k (Korn et al., 2009). That is $l(i) = H(d(i_1), d(i_2), \cdots, d(i_{d(i)}))$ for any node i, where $i_1, i_2, \cdots, i_{d(i)}$ are the neighbors of node i, and $d(i_1), d(i_2), \cdots, d(i_{d(i)})$ are the degrees of these neighbors, respectively. Degree centrality (Freeman, 1979) is the simplest measure of node centrality, using only the local structure around nodes. The degree of a node i, denoted as $d(i)$, is the number of links of node i in simple networks. Based on the lobby index, Lü, Zhou et al. (2016) proposed the node’s h-index, which uses more network structure information, to measure a node’s importance in simple networks. The 1-order h-index is just the lobby index, and the n-order h-index $h_i^{(n)} = H(h_{i_1}^{(n-1)}, h_{i_2}^{(n-1)}, \cdots, h_{i_{d(i)}}^{(n-1)})$ for any node i and any $n > 1$. For any node i, the degree, n-order h-index and coreness (Kitsak et al., 2010) are the initial, intermediate and steady state of the sequence $\{h_i^{(n)}\}_{n=0}^{\infty}$, respectively. The node’s h-index is a good tradeoff that in many cases can better quantify its importance than either degree or coreness (Lü, Zhou et al., 2016).

Some h-type centralities are defined on weighted networks. Zhao et al. (2011) extended the lobby index into the w-lobby index in the weighted network without considering the influence of the link strength (or weight) of the target node. In w-lobby index, node strength (Barrat, Barthélemy, Pastor-Satorras, & Vespignani, 2004) is used instead of node degree. For any node i, the node strength is defined as the summation of its link weights, namely, $s(i) = \sum_j s_{ij}$, where $s_{ij} \neq 0$ if node i and node j are connected with link weight s_{ij}, and $s_{ij} = 0$ otherwise. The strength of a node measures the importance of its links, but cannot distinguish the number of links. Zhao et al. (2011) proposed the h-degree in the weighted network. The h-degree $d_h(i)$ of node i is equal to k if k is the largest natural number such that i has at least k links with each strength at least equal to k (Zhao et al., 2011), that is, $d_h(i) = H(s_{ii_1}, s_{ii_2}, \cdots, s_{ii_{d(i)}})$. Schubert (2012) proposed the partnership ability index to measure jazz musicians’ collaboration, which can fit into the h-degree framework (Rousseau, 2012). The h-degree merely considers node degree and link weight without considering the influence of the neighboring nodes. Lü, Zhou et al. (2016) extended the node’s h-index to weighted networks and worked out an iterative process to calculate the weighted h-index, noting that the process should converge to s-core (Eidsaa & Almaas, 2013). However, the weighted h-index cannot distinguish the influence of the number of neighboring nodes on the importance of the target node. There are also some h-type centralities for weighted networks, such as c-index (Yan et al., 2013) and communication centrality (Zhai et al., 2013). The c-index $c(i)$ of any node i is the biggest integer k such that node i has at least k neighboring nodes satisfying that the product of each node strength and the link weight linked with node i is no less than k (Yan et al., 2013), that is, $c(i) = H(s_{ii_1}s(i_1), s_{ii_2}s(i_2), \cdots, s_{ii_{d(i)}}s(i_{d(i)}))$. Furthermore, the iterative c-index$c^{(n)}(i) = H(s_{ii_1}c^{(n-1)}(i_1), s_{ii_2}c^{(n-1)}(i_2), \cdots, s_{ii_{d(i)}}c^{(n-1)}(i_{d(i)}))$ for $n \geq 1$ and $c^{(0)}(i) = s(i)$, for any node i (Yan et al., 2013). According to their definitions, the c-index, lobby index and 1-order h-index of a node are the same in unweighted networks.

The arcs in directed networks can be expressed as “in” or “out” for a node. For a pair of linked nodes, when the node is the end of the relationship, this arc is the in-arc of the node. Conversely, if the node is the starting point of the relationship, this arc is the out-arc (Newman, 2010; Zhao et al., 2011). Inspired by this partition of arcs as in-arcs or out-arcs, in directed networks the degree can be divided into in-degree and out-degree, respectively (Newman, 2010). For any node i, the in-degree $d_{in}(i)$ is the number of in-arcs of node i, while out-degree $d_{out}(i)$ counts the out-arcs (Newman, 2010; Wasserman, 1994). Similar to directed degree centrality, many h-type centralities are extended to directed networks. Zhai et al. (2014) extended the lobby index to directed networks and proposed in-lobby index and out-lobby index. Zhao et al. (2012) introduced the directed h-degree for weighted directed networks. For any node i, the in-h-degree $h_I(i) = H(s_{k_1 i}, s_{k_2 i}, \cdots, s_{k_{d_{in}(i)} i})$ and out-h-degree $h_O(i) = H(s_{ij_1}, s_{ij_2}, \cdots, s_{ij_{d_{out}(i)}})$, where $W = \{s_{ij}\}$ is the adjacency matrix of the weighted directed network, $j_1, j_2, \cdots j_{d_{out}(i)}$ denote the neighbors directed from node i, and $k_1, k_2, \cdots k_{d_{in}(i)}$ denote the neighbors pointing to node i. Based on the node’s h-index, Lü, Zhou et al. (2016) defined the in-h-index and out-h-index. The n-order in-h-index $in - h_i^{(n)} = H(in - h_{k_1}^{(n-1)}, in - h_{k_2}^{(n-1)}, \cdots, in - h_{k_{d_{in}(i)}}^{(n-1)})$ and n-order out-h-index $out - h_i^{(n)} = H(out - h_{j_1}^{(n-1)}, out - h_{j_2}^{(n-1)}, \cdots, out - h_{j_{d_{out}(i)}}^{(n-1)})$ for any node i and any $n > 0$. The 0-order in-h-index is the in-degree and 0-order out-h-index is the out-degree.

These extensions of h-type centralities to directed networks only consider the direction of the arc and do not take into account the mutually reinforcing relationship of two reverse relations. However, this is an important factor that will affect the node’s importance in the directed network.

2.3. HITS algorithm

The Hypertext Induced Topic Selection (HITS) algorithm (Kleinberg, 1999) was developed to predict importance of webpages and became a famous algorithm for node centrality in directed networks (Ding et al., 2003). In directed networks, the arc directions usually characterize two kinds of reverse relations, e.g., cited or citing and information acquisition or information diffusion (Zhao & Ye, 2012). The HITS algorithm considers two roles of each node in directed networks, namely Authority and Hub (Kleinberg, 1999).

In the World Wide Web, the authoritative webpages are always reliable and provide original information on specific topics, while the hub webpages are those that link to many related authorities. Hubs and authorities exhibit a *mutually reinforcing relationship*: a good authority is pointed to by many good hubs and a good hub points to many good authorities (Kleinberg, 1999; Lempel & Moran, 2000). In a directed network, the authority score of a node equals the summation of the hub scores of all the nodes that point to this node, while the hub score of a node equals the summation of the authority scores of all the nodes being pointed at by this node. In the beginning of iteration, the hub scores of all nodes are assigned as 1.

Casting aside the background of webpage ranking, in the HITS algorithm the importance of the out-direction of the node is influenced by the importance of the in-direction of the neighboring nodes it points to, and the importance of the in-direction of the node is impacted by the importance of the out-direction of the neighboring nodes directed to it. Therefore, the HITS algorithm has been used to identify the importance of nodes against different backdrops, such as email networks (Campbell, Maglio, Cozzi, & Dom, 2003), citation networks (Liu & Lin, 2007), Q&A networks (Jurczyk & Agichtein, 2007), academic collaboration networks (Zhou, Zhang, & Cheng, 2014) and identification of super spreaders (Zhang, Chen, Dong, & Zhao, 2016).

2.4. Other centralities in weighted and/or directed networks

As one of the main focuses of network science and related fields, the research on node centralities is very rich. In addition to the above h-type centralities and HITS algorithm, many centrality measures are also defined in weighted and/or directed networks. In this subsection, we summarize some of their representative centrality measures.

Many centralities in weighted and/or directed networks are natural extensions of the methods originally designed for unweighted and/or undirected networks. The most common way to extend the centralities to directed networks is to take into account the out-arcs or in-arcs, respectively. Examples include the in-degree and out-degree, the in-h-degree and the out-h-degree, the in-h-index and out-h-index, and in-closeness and out-closeness. Similarly, in a weighted directed network of n nodes, the node strength (Barrat et al., 2004) can also be defined as the in-strength and out-strength. For any node i, its in-strength (out-strength) is defined as the summation of its in-arc (out-arc) weights.

The extension of centralities to weighted networks is more complicated than the extension to directed networks. Closeness centrality and betweenness centrality are classic path-based centralities. The key to extending closeness and betweenness to weighted networks is redefining the shortest paths. Different from the distance of a link in unweighted networks, the distance of a weighted link is related to its weight. Using Dijkstra's (1959) algorithm, Newman (2001) and Brandes (2001) proposed to adopt the reciprocal of weights to extend closeness and betweenness, respectively. Kitsak et al. (2010) suggested that coreness is a better indicator of a node's influence on spreading dynamics than degree centrality. The coreness of a node is measured by k-core/k-shell decomposition (Bollobás, 1984; Carmi, Havlin, Kirkpatrick, Shavitt, & Shir, 2006; Dorogovtsev, Goltsev, & Mendes, 2006; Seidman, 1983). The coreness calculation requires all of the network information and has a higher cost than calculating node degree and node's h-index (Lü, Zhou et al., 2016). The classic k-shell decomposition can be extended to weighted networks by replacing the node degree with the weighted degree in the decomposition process. Eidsaa and Almaas (2013) proposed s-shell/s-core decomposition, which takes node strength instead of node degree. Garas, Schweitzer, and Havlin (2012) extended k-shell decomposition with the weighted degree proposed by Opsahl et al. (2010). The weighted degree can integrate node degree and node strength in a nonlinear way. It can also be defined in a linear method based on degree and node strength (Wei, Liu, Wei, Gao, & Deng, 2015).

PageRank (Brin & Page, 1998) is a famous webpage ranking algorithm and is used to rank webpages in Google search. It supposes that the importance of a webpage is determined by both the number and the quality of pages linked to it. A webpage that is linked to by many pages with high PageRank value will receive a high rank itself. At present, PageRank has been applied to rank a broad range of objects via their directed network structure (Chen, Xie, Maslov, & Redner, 2007; Gleich, 2015). The extension of PageRank to weighted networks is simple and clear. In each step, the PageRank value of a node will be distributed to its outgoing neighbors according to the arc weights (Ding, Yan, Frazho, & Caverlee, 2009; Xing & Ghorbani, 2004). PageRank has many variations. For example, LeaderRank (Lü, Zhang, Ho, & Zhou, 2011) addresses the use of PageRank in disconnected networks, and it is further generalized to weighted networks by Li, Zhou, Lü, and Chen (2013). In addition, there are some studies of node centrality in weighted and/or directed networks, such as the weighted LocalRank for weighted networks, ClusterRank (Chen, Lü, Shang, Zhang, & Zhou, 2012) for directed networks (Gao, Wei, Hu, Mahadevan, & Deng, 2013), rethinking centrality (Stephenson & Zelen, 1989) for weighted networks, and the SALSA algorithm (Lempel & Moran, 2000) for directed networks.

Because of the complexity of weighted directed networks, there are fewer studies about node centrality in weighted directed networks, but both theory and practice have greater demand for it. In this paper, we propose an extension of the

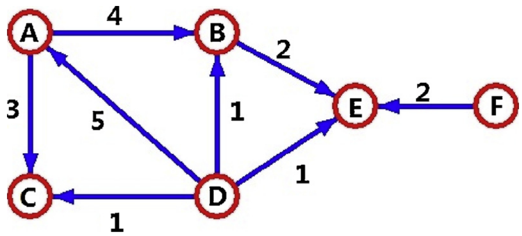

Fig. 1. An example of a weighted directed network.

h-index to quantify a node's importance in weighted directed networks, in which the mutually reinforcing relationship of two reverse relations will be taken into account. In the next section, we will give a specific theoretical analysis of the method, including its definition, properties and algorithms.

3. Theoretical aspects

3.1. Bi-directional h-index: definition and properties

Suppose there are no loops or multiple arcs for the weighted directed networks discussed in this paper. Based on the HITS algorithm and h-index, we propose the following concept of bi-directional h-index.

Definition 1 *((Bi-directional h-index)).* $G(V, E)$ denotes a weighted directed network, where V is the set of nodes and E is the set of arcs. The arc weight directed from any node i to any node j is denoted by s_{ij}. The in-degree and out-degree of any node i are denoted by $d_{in}(i)$ and $d_{out}(i)$, respectively. Then node i has $d_{out}(i)$ neighbors directed from it denoted by $j_1, j_2, \cdots j_{d_{out}(i)}$ and has $d_{in}(i)$ neighbors denoted by $k_1, k_2, \cdots k_{d_{in}(i)}$ directed to node i. Then, we define the bi-directional h_{in}-index and h_{out}-index of node i as,

$$h_{in}(i) = H(s_{k_1 i} d_{out}(k_1), s_{k_2 i} d_{out}(k_2), \cdots, s_{k_{d_{in}(i)} i} d_{out}(k_{d_{in}(i)})), \tag{1}$$

$$h_{out}(i) = H(s_{ij_1} d_{in}(j_1), s_{ij_2} d_{in}(j_2), \cdots, s_{ij_{d_{out}(i)}} d_{in}(j_{d_{out}(i)})). \tag{2}$$

In other words, node i's h_{in}-index (h_{out}-index) is the h-index of the products of the in-arc weights (out-arc weights) of node i and the out-degrees (in-degree) of the corresponding neighbors. "H-index" here refers to the operator H. A high h_{in}-index (h_{out}-index) implies that the node not only has high in-degree (out-degree), but also maintains relatively high in-arc weights (out-arc weights) or the linked nodes that point to it have many out-arcs (in-arcs). Obviously, the bi-directional h_{in}-index of node i is the non-decreasing function of the out-degree of the neighboring nodes directed to it, the in-arc weights and the in-degree of node i. Meanwhile, h_{out}-index of node i refers to the non-decreasing function of the in-degree of the neighboring nodes it points to, the out-arc weights and the out-degree of node i. Consequently, bi-directional h-index realizes the effective tradeoff of the three important factors influencing the importance of the nodes. In particular, in the case of an undirected network, the link can be regarded as bi-directional, and the h_{in}-index and h_{out}-index are of equal value. Furthermore, we can suppose the link weight of the unweighted undirected network is 1 and all the links are bi-directional. Then h_{in}-index is equal to h_{out}-index, 1-order h-index, lobby index and c-index.

Next, in Fig. 1 we demonstrate the calculation of the bi-directional h-index with nodes D and E. Node D points to its neighbors A, B, C and E with the weights of 5, 1, 1 and 1, and their in-degrees are 1, 2, 2 and 3, respectively. Then for node D the bi-directional h_{out}-index $h_{out}(D) = H(5 \times 1, 1 \times 2, 1 \times 2, 1 \times 3) = H(5,2,2,3) = 2$. Node E is pointed to by nodes B, D and F with weights of 2, 1 and 2, and their out-degrees are 1, 4 and 1, respectively. Then for node E the bi-directional h_{in}-index $h_{in}(E) = H(2 \times 1, 1 \times 4, 2 \times 1) = H(2, 4, 2) = 2$.

There are some simple properties concerning the bi-directional h-index.

Property 1. For any node i in a weighted directed network $G(V, E)$, if the network size N is finite, then

(1) $0 \le h_{in}(i) \le d_{in}(i) \le N-1$ and $0 \le h_{out}(i) \le d_{out}(i) \le N-1$;
(2) If the arc weight is a positive integer, so that the link strength is at least 1, then if $d_{in}(i) \neq 0$, we have $h_{in}(i) \ge 1$, and if $d_{out}(i) \neq 0$, then $h_{out}(i) \ge 1$.

The proof of Property 1 is shown in Appendix A1.

Part (1) of Property 1 indicates that the bi-directional h-index is nonnegative, h_{in}-index takes the in-degree as the upper bound and h_{out}-index takes the out-degree as the upper bound. For this reason, the h_{in}-index (h_{out}-index) of a node is 0 if its in-degree (out-degree) is 0. For the isolated node i, $h_{in}(i) = h_{out}(i) = 0$. It manifests that the bi-directional h-index has a smaller scope of change and a lower node discrimination than degree centrality. Nonetheless, the low discrimination will not influence the identification of the set of most important nodes. Besides, it is more difficult for a node to acquire a high-value

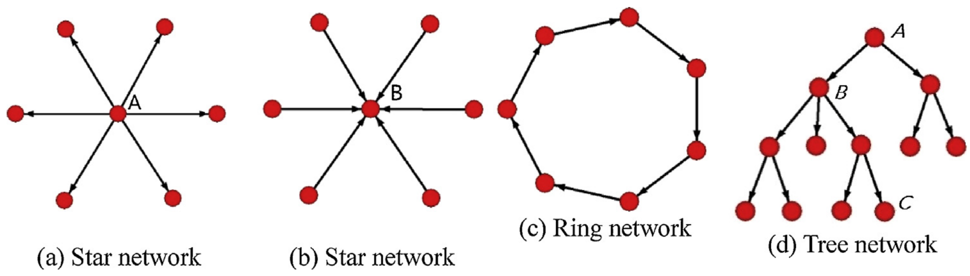

Fig. 2. Some particular weighted directed networks.

bi-directional h-index than to have a high node degree, because it is not merely influenced by the number of arcs but also impacted by its arc weights and the importance of its neighbors, which are more difficult to control.

The link weights in weighted networks are usually positive integers, for example, co-author networks with the number of collaborations as the link weights, microblog message forwarding networks with the number of forwardings as the link weights and journal citation networks based on the citations. Part (2) of Property 1 demonstrates that when the arc weight is a positive integer, and if the node has in-arcs (out-arcs), the h_{in}-index (h_{out}-index) is at least 1. So, in a weighted directed network the arc weight is a positive integer, $h_{in}(i) = 1$ ($h_{out}(i) = 1$) if the in-degree (out-degree) of node i is 1.

Property 2. We describe the properties of the bi-directional h-index in some particular weighted directed networks.

(1) In a star network similar to Fig. 2(a) and (b), the h_{out}-index of node A is of equal value with its out-h-degree, and the h_{in}-index of node B is of equal value with its in-h-degree.
(2) Since the out-degree and the in-degree of any node i in the directed ring network indicated in Fig. 2(c) are both 1, its bi-directional h-index is either 0 or 1. If the in-arc (out-arc) weight is less than 1, then h_{in}-index (h_{out}-index) is 0, otherwise, it is 1. Specifically, when the arc weight is a positive integer, $h_{in}(i) = h_{out}(i) = 1$.
(3) If node i in the tree network illustrated in Fig. 2(d) is the root node (e.g., node A) or the nonleaf node (e.g., node B), then h_{out}-index is of equal value with the out-h-degree; when node i is a nonleaf node or leaf node (e.g., node C), then $h_{in}(i) = 0$ if the in-arc weight of node i is less than 1, else $h_{in}(i) = 1$; if node i is the root node, then $h_{in}(i) = 0$; if node i is a leaf node, then $h_{out}(i) = 0$.

Details of the directed h-degree (Zhao & Ye, 2012) are given in Subsection 2.2. The proof of Property 2 is shown in Appendix A2.

3.2. Synchronous updating process of bi-directional h-indices and their limits

The bi-directional h-index has self-referential characteristics (Jackson, 2008). It calculates the importance of a node on the prerequisite that the importance of its neighboring nodes is determined. Many node centrality measures are established on this basis, such as Katz prestige (Katz, 1953), eigenvector centricity (P. Bonacich, 1972), PageRank algorithm (Brin & Page, 1998) and HITS algorithm (Kleinberg, 1999).

The above-defined bi-directional h-index exhibits the importance of the neighboring nodes through the node degree. It is a local heuristic method to measure the importance of the nodes, and the calculation of the bi-directional h-index of each node makes use of the information of the two-step local network with the node as the center. To more accurately measure the importance of the neighboring nodes, we extend the bi-directional h-index to its sequence form. We adopt the mutually reinforcing method to constantly calculate the value of the bi-directional h-index in an effort to update the importance of the nodes.

In a weighted directed network, the mutually reinforcing relationship can be proposed as the following general operations. We define $h_{in}^{(0)}(i) = d_{in}(i)$ and $h_{out}^{(0)}(i) = d_{out}(i)$ to be the 0-order bi-directional h-index of any node i, and define the n-order bi-directional h-index ($n > 0$) iteratively as,

$$h_{in}^{(n)}(i) = H(s_{k_1 i} h_{out}^{(n-1)}(k_1), s_{k_2 i} h_{out}^{(n-1)}(k_2), \cdots, s_{k_{d_{in}(i)} i} h_{out}^{(n-1)}(k_{d_{in}(i)})), \tag{3}$$

$$h_{out}^{(n)}(i) = H(s_{ij_1} h_{in}^{(n-1)}(j_1), s_{ij_2} h_{in}^{(n-1)}(j_2), \cdots, s_{ij_{d_{out}(i)}} h_{in}^{(n-1)}(j_{d_{out}(i)})). \tag{4}$$

The rest of the symbols in formulas (3) and (4) have the same meaning as in Definition 1. The bi-directional h-index in Definition 1 is the 1-order bi-directional h-index, namely $h_{in}^{(1)}(i) = h_{in}(i)$ and $h_{out}^{(1)}(i) = h_{out}(i)$. For simplicity, n-order bi-directional h-index ($n \geq 0$) are collectively referred to as bi-directional h-index hereafter.

Formulas (3) and (4) make use of operator H to express the mutually reinforcing relationship of h_{in}-index and h_{out}-index. Since the bi-directional h-index of all the nodes are updated simultaneously in each step of the iterative process, we call this process the "synchronous updating process." The sequences $h_{in}^{(0)}(i), h_{in}^{(1)}(i), h_{in}^{(2)}(i), \cdots$ and $h_{out}^{(0)}(i), h_{out}^{(1)}(i), h_{out}^{(2)}(i), \cdots$ are formed in the iterative process, for any node *i*. Next, we will analyze the stability of this updating process.

Theorem 1. *For any node i in a weighted directed network G(V, E), its bi-directional h-index sequences* $h_{in}^{(0)}(i), h_{in}^{(1)}(i), h_{in}^{(2)}(i), \cdots$ *and* $h_{out}^{(0)}(i), h_{out}^{(1)}(i), h_{out}^{(2)}(i), \cdots$ *are non-increasing and bounded, respectively. Therefore, in finite steps* n_∞ *the two sequences converge respectively, and the limits are denoted as* $h_{in}^{\infty}(i)$ *and* $h_{out}^{\infty}(i)$*, namely,*

$$\lim_{n\to\infty} h_{in}^{(n)}(i) = h_{in}^{(n_\infty)}(i) = h_{in}^{\infty}(i), \tag{5}$$

$$\lim_{n\to\infty} h_{out}^{(n)}(i) = h_{out}^{(n_\infty)}(i) = h_{out}^{\infty}(i). \tag{6}$$

The proof of Theorem 1 is shown in Appendix A3.

By Theorem 1, if the network size is finite, then in finite steps the bi-directional h-index sequences converge for all nodes. In the stable state, for any node *i* formulas (3) and (4) can be represented as,

$$h_{in}^{\infty}(i) = H(s_{k_1 i} h_{out}^{\infty}(k_1), s_{k_2 i} h_{out}^{\infty}(k_2), \cdots, s_{k_{d_{in}(i)} i} h_{out}^{\infty}(k_{d_{in}(i)})), \tag{7}$$

$$h_{out}^{\infty}(i) = H(s_{ij_1} h_{in}^{\infty}(j_1), s_{ij_2} h_{in}^{\infty}(j_2), \cdots, s_{ij_{d_{out}(i)}} h_{in}^{\infty}(j_{d_{out}(i)})). \tag{8}$$

Namely, the bi-directional h-index of all the nodes will no longer change with iterative updates. Hence, synchronous updating provides a series of indices to measure the node centrality. In addition to the directed degree centrality, all other *n*-order bi-directional h-index ($n \geq 1$) and their limits can also be considered as centrality measures. With the increasing of the iterations, the network information used in the calculation increases, and h_{in}^{∞}-index and h_{out}^{∞}-index use the most information. They are the steady state values of node importance driven by operator *H*.

Property 3. We have the following properties relating to the sequences of bi-directional h-index.

(1) For any node *i* in a weighted directed network *G*(*V*, *E*), if the arc weight is a positive integer, then if $d_{in}(i) \neq 0$, we have

$$1 \leq h_{in}^{\infty}(i) \leq \cdots \leq h_{in}^{(2)}(i) \leq h_{in}^{(1)}(i) \leq h_{in}^{(0)}(i) = d_{in}(i) \leq N-1,$$

and if $d_{out}(i) \neq 0$, then

$$1 \leq h_{out}^{\infty}(i) \leq \cdots \leq h_{out}^{(2)}(i) \leq h_{out}^{(1)}(i) \leq h_{out}^{(0)}(i) = d_{out}(i) \leq N-1.$$

Specifically, if $d_{in}(i) = 1$, then $h_{in}^{(n)}(i) = h_{in}^{\infty}(i) = 1 (n \geq 0)$; if $d_{out}(i) = 1$, then $h_{out}^{(n)}(i) = h_{out}^{\infty}(i) = 1 (n \geq 0)$.

(2) Take any link in undirected networks as bi-directional, then for any node *i* in a given undirected network, $h_{in}^{(n)}(i) = h_{out}^{(n)}(i) (n \geq 0)$ and $h_{in}^{\infty}(i) = h_{out}^{\infty}(i)$.
(3) Take unweighted networks as weighted networks whose arc weights are all 1, then in unweighted undirected networks the bi-directional h-index is consistent with the node's h-index and iterative c-index; the 1-order bi-directional h-index is consistent with the lobby index and c-index.

The details of the node's h-index, c-index, iterative c-index and lobby index are given in Subsection 2.2. It is easy to prove Property 3 according to Property 1 and Theorem 1; thus, it is omitted here. Theorem 1 and Property 3 show the value range of bi-directional h-index sequences. In Section 4, we will further analyze the distribution and convergence speed of bi-directional h-index through a real example. Property 3(3) indicates that bi-directional h-index covers the present concept of lobby index, node's h-index, c-index and iterative c-index. It establishes a greater concept framework that incorporates all of them. Therefore, bi-directional h-index is feasible and meaningful.

3.3. Asynchronous updating process

The synchronous updating rapidly converges to the limits h_{in}^{∞}-index and h_{out}^{∞}-index. However, the updating process from $h_{out}^{(n-1)}$ and $h_{in}^{(n-1)}$ to $h_{out}^{(n)}$ and $h_{in}^{(n)}$ requires updating of all the nodes at the same time step according to formulas (3) and (4). Thus, in principle, it requires a centralized controller to set up a global clock that records the order *n*, and the waiting time of computer processor in the synchronous updating process increases with the increasing of the network size at each time step. In addition, if the target network is evolving, the addition of a single link will require recalculating all the sequences. This limits the application of h_{in}^{∞}-index and h_{out}^{∞}-index to large-scale networks and dynamic networks. Fortunately, asynchronous updating can be used to deal with this problem. Lü, Zhou et al. (2016) proposed a novel asynchronous algorithm to calculate the coreness, which was driven by operator *H* and randomly selected a node to update at each iteration. Lee and Zhou (2017) improved the above algorithm and proposed two algorithms to select nodes and update their intermediate values towards the coreness, which can help in accelerating the convergence of the coreness calculation in asynchronous updating. In this paper, we propose an asynchronous updating process to calculate h_{in}^{∞}-index and h_{out}^{∞}-index. The asynchronous updating process is described as follows.

Given a weighted directed network $G(V, E)$, we define $g_{in}(v) = d_{in}(v)$ and $g_{out}(v) = d_{out}(v)$ for every node v. In each iteration of the asynchronous updating process, a set S of nodes is randomly selected, and then the $g_{in}(i)$ and $g_{out}(i)$ are updated for any node i in S. That is,

$$H(s_{k_1 i} g_{out}(k_1), s_{k_2 i} g_{out}(k_2), \cdots, s_{k_{d_{in}(i)} i} g_{out}(k_{d_{in}(i)})) \rightarrow g_{in}(i), \quad (9)$$

$$H(s_{ij_1} g_{in}(j_1), s_{ij_2} g_{in}(j_2), \cdots, s_{ij_{d_{out}(i)}} g_{in}(j_{d_{out}(i)})) \rightarrow g_{out}(i). \quad (10)$$

The rest of the symbols in formulas (9) and (10) have the same meaning as in Definition 1.

Theorem 2. *Given a weighted directed network G(V, E), if network size is finite, the updating processes by formulas (9) and (10) will respectively reach steady states* $(g_{in}^{\infty}(1), g_{in}^{\infty}(2), \ldots, g_{in}^{\infty}(N))$ *and* $(g_{out}^{\infty}(1), g_{out}^{\infty}(2), \ldots, g_{out}^{\infty}(N))$ *after finite iterations, namely, for any node i,*

$$g_{in}^{\infty}(i) = H(s_{k_1 i} g_{out}^{\infty}(k_1), s_{k_2 i} g_{out}^{\infty}(k_2), \cdots, s_{k_{d_{in}(i)} i} g_{out}^{\infty}(k_{d_{in}(i)})), \quad (11)$$

$$g_{out}^{\infty}(i) = H(s_{ij_1} g_{in}^{\infty}(j_1), s_{ij_2} g_{in}^{\infty}(j_2), \cdots, s_{ij_{d_{out}(i)}} g_{in}^{\infty}(j_{d_{out}(i)})), \quad (12)$$

and $g_{in}^{\infty}(i) = h_{in}^{\infty}(i)$, $g_{out}^{\infty}(i) = h_{out}^{\infty}(i)$.

The proof of Theorem 2 is shown in Appendix A4.

Note that formulas (9) and (10) neglect the time label and present the rules for updating by assignment. Theorem 2 shows the updating of the g value of each node in set S of arbitrarily selected nodes at the current time. The new g value should be calculated according to the g value of each node in the network of the previous time according to formulas (9) and (10). The number of nodes in set S is arbitrary and may be different at different iterative steps. The asynchronous updating process does not update the bi-directional h-index of all the nodes each time; therefore, for any node i, before it reaches the steady state, all the intermediate g values are not necessarily equal to any order of bi-directional h-index. Asynchronous updating provides a flexible and more operable way to apply the bi-directional h-index in large-scale networks and dynamic networks. It can overcome disadvantages of the synchronous updating process and realizes the calculations of the limits of n-order bi-directional h-index ($n \geq 0$) in the sporadic, local updating process. In large-scale networks, the asynchronous updating process also helps to exploit parallelism in computing to achieve higher performance, such as shortening the waiting time of processors. In dynamic networks, it can be calculated on the basis of the previous calculations, without requiring all nodes to restart the calculation again.

4. An empirical study

4.1. Dataset

As a vital information network, the journal citation network takes advantage of the citation relationship between journals to depict the information flow between them. It is a typical weighted directed network. Without loss of generality, we explore the empirical nature of the bi-directional h-index in a real journal citation network. We selected 24 leading business journals as the data source, which are used by UT Dallas' Naveen Jindal School of Management to provide the top 100 business school rankings (see Appendix A5).

To construct the network, we collected publications and their references of these journals from 2000 to 2009 from the Web of Science database. These publications are required to be labeled as "Article" in the database. The 24 journals are taken as the nodes of the network. For journal A and journal B, if journal A cites journal B, then there is an arc from A to B. The citation relationship between journals was constructed by extracting citations from articles and their references. There is no limitation on the years of references. Suppose that journal A contains N_A articles, and for any article i in journal A, $n_i(A,B)$ denotes the number of articles in journal B cited by article i. Then the arc weight $s_{AB} = \sum_{i=1}^{N_A} n_i(A, B)$ in the journal citation network. No loops (the arc starting from a node and directed to itself) are included in the network, i.e., self-citation is excluded. The journal citation network reflects the citation relationship between the 24 journals in the period 2000–2009.

In the journal citation network, the journals mainly absorb information or knowledge from other journals (Zhao & Ye, 2012). Naturally, a journal boasting a favorable ability to absorb information and knowledge will massively and frequently cite information and knowledge from the highly influential journals. Likewise, highly influential journals will be massively and frequently cited by journals boasting a strong ability to absorb information. The journals of high absorbing ability and influence stimulate the transmission of information and knowledge between journals. According to the structure of the network and the concept of the bi-directional h-index, the h_{out}-index in the journal citation network depicts the ability of the journals to absorb information and knowledge, while the h_{in}-index indicates the influence of the information and knowledge in the journals.

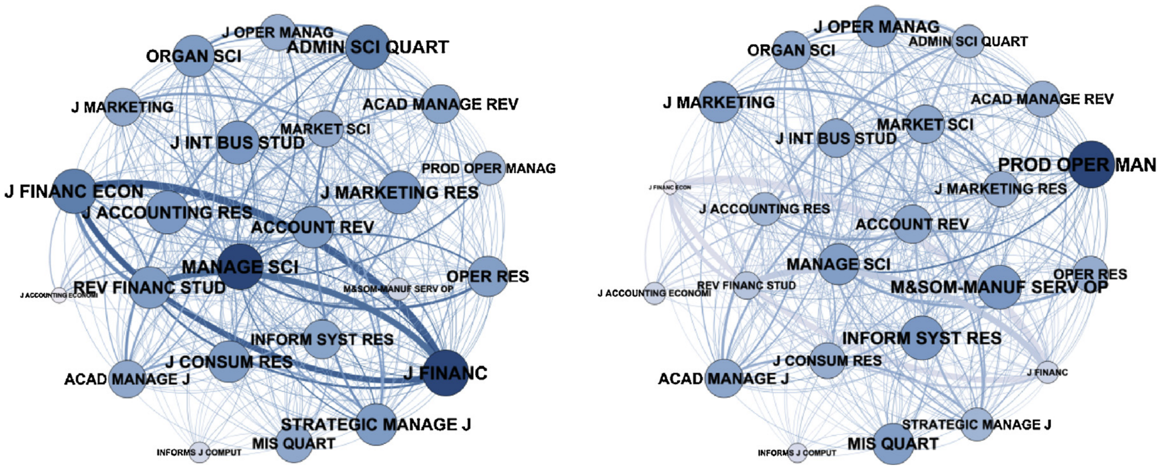

Fig. 3. Journal citation network (In the left and the right of the figure, the size and color depth of nodes are proportional to $h_{in}^{(2)}$-index and $h_{out}^{(2)}$-index, respectively. Arc direction is clockwise from the source node to the target node.).

4.2. Results and analysis

The n-order bi-directional h-index ($n \geq 0$) and its limits of all the nodes are calculated in the journal citation network. When $n = 2$, the bi-directional h-index of all nodes converge, namely, $h_{in}^{(2)}(i) = h_{in}^{\infty}(i)$ and $h_{out}^{(2)}(i) = h_{out}^{\infty}(i)$ for any node i. As far as this case is concerned, calculating the limits of the bi-directional h-index only utilizes the local network information that is no more than three steps away from it for every node i. The absolute deviations of two successive iterations are $d_n(h_{in}) = \sum_{i \in V} |h_{in}^{(n)}(i) - h_{in}^{(n-1)}(i)|$ and $d_n(h_{out}) = \sum_{i \in V} |h_{out}^{(n)}(i) - h_{out}^{(n-1)}(i)|$ ($n \geq 1$). When $n = 1$, $d_1(h_{in}) = 19$ and $d_1(h_{out}) = 11$; when $n = 2$, $d_2(h_{in}) = 1$ and $d_2(h_{out}) = 1$; and when $n = 3$, $d_3(h_{in}) = d_3(h_{out}) = 0$. It thus can be seen that the absolute deviations rapidly drop with the increasing of n. The bi-directional h-index of a node is influenced by other nodes in the network, but the influence is weaker with longer distance. In this case, the stable value of the bi-directional h-index of most nodes (23 out of 24) is influenced chiefly by their neighboring nodes one step away, and $h_{in}^{(1)}$-index and $h_{out}^{(1)}$-index have reached their stable limits. The 1-order bi-directional h-index is a better choice in this example, because it only undergoes the operation of operator H once, without needing to master the information of the whole network, and the difference between it and the stable value of bi-directional h-index is very small.

Fig. 3 demonstrates the journal citation network. The two charts in Fig. 3 show that the node's $h_{in}^{(2)}$-index and $h_{out}^{(2)}$-index are sharply different. Further, we show the scatter plot in Fig. 4 to analyze their relationship. Since the full names of some journals are too long, we use the abbreviated names of journals in Figs. 3 and 4, which can be seen in Appendix A5. For any journal, the difference of $h_{in}^{(2)}$-index and $h_{out}^{(2)}$-index can reflect the difference of the two sorts of node's importance, which boasts great significance for analyzing the position of the node in the network (Zhao & Ye, 2012). Some journals, such as *Journal of Finance* and *Journal of Financial Economics*, boast higher $h_{in}^{(2)}$-index but lower $h_{out}^{(2)}$-index. Such journals exhibit stronger influence but weaker absorbing ability. *Manufacturing and Service Operations Management* is just the opposite, boasting a lower $h_{in}^{(2)}$-index but a higher $h_{out}^{(2)}$-index. Such journals are competent in absorbing information and knowledge but have weaker influence. The journals with higher $h_{in}^{(2)}$-index and $h_{out}^{(2)}$-index are naturally the most important ones in the journal citation network. For instance, *Management Science* is concurrently more competent in journal influence and ability to absorb information and knowledge, thus it tremendously expedites the exchanges and transmission between different journals and even between different disciplines.

In order to look at consistency of centrality measures, we calculate the Kendall correlation coefficient (Kendall, 1938) between some common centrality measures and the bi-directional h-index in the journal citation network. The correlation analysis can display whether different measures are replaceable. A strong correlation means the two are strongly replaceable; medium correlation indicates that the two cannot be replaced but reflect each other to some extent (Korn et al., 2009). These centrality measures include directed degree centrality (including in-degree and out-degree), directed node strength (including in-strength and out-strength), directed h-degree (including in-h-degree and out-h-degree) and HITS algorithm (including Authority score and Hub score). These measures are defined in Section 2. The $h_{in}^{(n)}$-index and $h_{out}^{(n)}$-index ($n = 1$ and 2) are used in correlation analysis.

Table 1 shows that the $h_{in}^{(1)}$-index ($h_{out}^{(2)}$-index) and the $h_{in}^{(2)}$-index ($h_{out}^{(2)}$-index) are strongly correlated, with the correlation coefficient reaching 0.986 (0.984), which further proves that although the 1-order bi-directional h-index makes use of the information in a two-step local network, it is largely approximate to the limit of the bi-directional h-index that utilizes the most network information. Therefore, the 1-order bi-directional h-index is a simple and optional index. The correlations

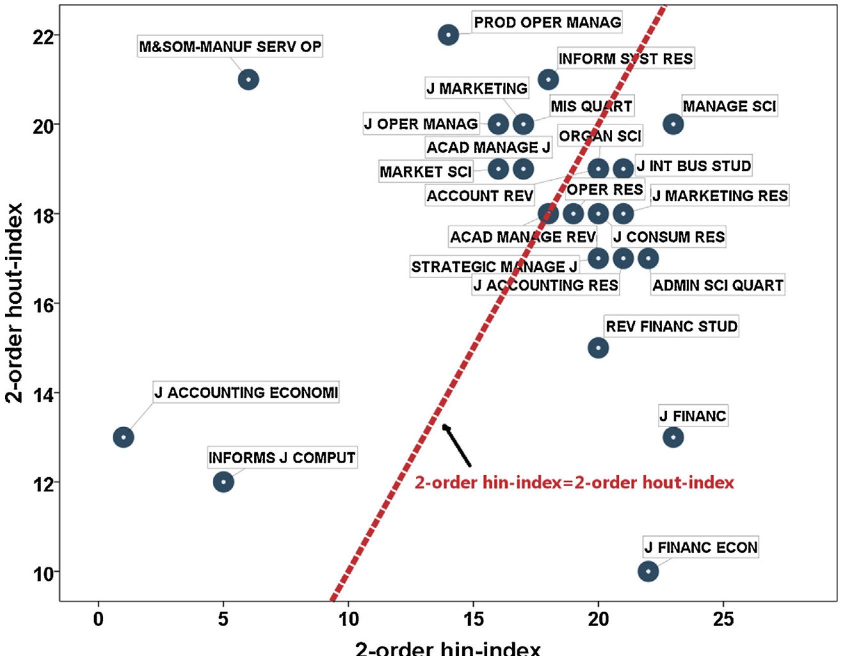

Fig. 4. The scatter plot of $h_{in}^{(2)}$-index and $h_{out}^{(2)}$-index.

Table 1
Kendall correlation coefficient of the bi-directional h-index and other centrality measures.

	$h_{in}^{(1)}$	$h_{out}^{(1)}$	$h_{in}^{(2)}$	$h_{out}^{(2)}$	d_{in}	d_{out}	s_{in}	s_{out}	h_I	h_O	*Authority*
$h_{out}^{(1)}$	−0.308	–									
$h_{in}^{(2)}$	0.986	−0.281	–								
$h_{out}^{(2)}$	−0.332	0.984	−0.305	–							
d_{in}	0.825	−0.316	0.799	−0.324	–						
d_{out}	−0.267	0.862	−0.240	0.875	−0.273	–					
s_{in}	0.672	−0.362	0.670	−0.347	0.635	−0.258	–				
s_{out}	0.274	−0.126	0.248	−0.126	0.319	−0.096	0.384	–			
h_I	0.636	−0.245	0.653	−0.234	0.600	−0.169	0.862	0.373	–		
h_O	−0.127	0.727	−0.108	0.731	−0.086	0.733	−0.107	0.154	0.008	–	
Authority	0.825	−0.316	0.799	−0.324	1.000	−0.273	0.635	0.319	0.600	−0.086	–
Hub	0.825	−0.316	0.799	−0.324	1.000	−0.273	0.635	0.319	0.600	−0.086	1.000

Note: d_{in} and d_{out} are in-degree and out-degree, respectively. s_{in} and s_{out} are in-strength and out-strength, respectively. h_I and h_O are in-h-degree and out-h-degree, respectively. *Authority* and *Hub* are Authority score and Hub score in HITS algorithm, respectively. $h_{in}^{(n)}$ and $h_{out}^{(n)}$ are $h_{in}^{(n)}$-index and $h_{out}^{(n)}$-index, respectively, $n = 1$ and 2.

are moderate between the $h_{in}^{(n)}$-index ($h_{out}^{(n)}$-index) and other all indices including in-degree (out-degree), in-strength (out-strength), in-h-degree (out-h-degree) and Authority score (Hub score), where $n = 1, 2$. This illustrates that the bi-directional h-index is different from the other centrality measures and irreplaceable but can reflect their influence on the importance of the nodes. And the measures in different directions are mostly uncorrelated or weakly correlated. This shows that it is reasonable for each directed index to depict the centrality of the nodes in different directions, reflecting the two completely different importance of the nodes.

5. Conclusion and discussion

In this work, we introduced the bi-directional h-index, which provides a family to measure the importance of nodes in weighted directed networks. The bi-directional h-index starts from the directed degree centrality that using the least network information, and iteratively uses the more extensive network information through operator H to describe the importance of the nodes until convergence. It provides the sequences of indices for measuring the importance of the node, namely, $h_{in}^{(0)}, h_{in}^{(1)}, h_{in}^{(2)}, \cdots, h_{in}^{\infty}$ and $h_{out}^{(0)}, h_{out}^{(1)}, h_{out}^{(2)}, \cdots, h_{out}^{\infty}$. The theoretical analysis displays that the bi-directional h-index sequences converge after finite iterations. It is feasible and significant to establish a greater conceptual framework that includes the concepts of lobby index, node's h-index, c-index and iterative c-index. The 1-order bi-directional h-index has a lower cost of calculation for large-scale networks. It uses the information of two-step local network with the target node as the center, and applies operator H only once. The analysis based on a journal citation network illustrates that the sequences converge at a high speed, and the 1-order bi-directional h-index can reach the approximate effects of the limit of the bi-directional h-index. The Kendall correlation analysis shows that the bi-directional h-index is different from directed degree centrality, directed node strength, directed h-degree and the HITS algorithm. It is irreplaceable and can reflect the influence of the above indices on the importance of the nodes. Moreover, the bi-directional h-index is universal for measuring node importance. For example, it can also be used to measure the influence of users in Q&A networks and identify the authoritative webpages and hub webpages in hyperlink networks.

With the aim of more flexibly applying the bi-directional h-index in large-scale networks and dynamic networks, we propose an asynchronous updating method to work out h_{in}^{∞}-index and h_{out}^{∞}-index. It can still guarantee the convergence to h_{in}^{∞}-index and h_{out}^{∞}-index within finite iterations through a decentralized local updating. However, randomly selecting the set of nodes to be updated in each iteration may greatly extend the time required before arriving at the steady states, even in static networks. Thus, the process for selecting which set of nodes to be updated is a nontrivial issue. For example, we can shorten the convergence time by reducing the selection probability of nodes that have been updated many times but have g values that seldom changed. In addition, the change of a node's g value will enhance the updating probabilities of its neighbors, which makes this issue more complicated and thus more interesting (Lü, Zhou et al., 2016).

As there is no "perfect" single measure (Zhao & Ye, 2012), the bi-directional h-index also has limitations. With the increasing of the orders, the nodes with lower bi-directional h-index are difficult to be distinguished because of their equal scores, although this has little impact on identifying the set of the most important nodes. To boost the discrimination of the nodes, a stepwise strategy can be established based on the bi-directional h-index sequences. Specifically, it places the nodes in the network in descending order according to priority from the limit to the 0-order of bi-directional h-index and takes the ordinal number as the ranking of their importance. Evidently, the discrimination performance of the score for the importance of the nodes gained in the strategy will be better than the limits of bi-directional h-index.

This paper supposes there are no multiple arcs in the network. That is to say, it has no more than one arc from any node A to any node B. As a matter of fact, a network with multiple arcs will usually be handled as one without multiple arcs according to the actual background. Multiple arcs can be considered as a single arc and are often defined to be the summation, the maximum or the minimum of the weight of the multiple arcs in the weighted network. Nonetheless, in some cases, the importance of the nodes in the network needs to be calculated with multiple arcs. Fortunately, the bi-directional h-index can be naturally extended to such networks with multiple arcs. For instance, in calculating of $h_{out}^{(n)}$-index ($n \geq 1$) for any node A, if there are more than two arcs from A to node B, the product of each out-arc weight and the $h_{in}^{(n-1)}$ of node B is calculated, and then the $h_{out}^{(n)}$-index value is calculated by using operator H. The $h_{out}^{(0)}$-index is the number of its out-arcs. The $h_{in}^{(n)}$-index is similarly defined. Under this definition it can be proven that Theorem 1 and Theorem 2 remain true.

Author contributions

Li Zhai: Conceived and designed the analysis; Collected the data; Contributed data or analysis tools; Performed the analysis; Wrote the paper.

Xiangbin Yan: Conceived and designed the analysis; Wrote the paper.

Guojing Zhang: Contributed data or analysis tools; Performed the analysis; Wrote the paper.

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Appendix A. Proof of Property 1

Appendix A1. Proof of Property 1

(1) By formula (1) and definition of operator H, we have $0 \leq h_{in}(i) \leq d_{in}(i)$. Because the network contains no loops, its maximum of in-degree is N-1 for any node i and $0 \leq h_{in}(i) \leq d_{in}(i) \leq N-1$. Similarly, $0 \leq h_{out}(i) \leq d_{out}(i) \leq N-1$ can be proved.

(2) If $d_{in}(i) \neq 0$, then there is at least one node j, making the arc weight $s_{ji} \geq 1$, so $s_{ji} d_{out}(j) \geq 1$, and $h_{in}(i) = H(s_{k_1 i} d_{out}(k_1), s_{k_2 i} d_{out}(k_2), \cdots, s_{k_{d_{in}(i)} i} d_{out}(k_{d_{in}(i)})) = \max\{y | s_{k_y i} d_{out}(k_y) \geq y\} \geq 1$. Similarly, we can also prove that if $d_{out}(i) \neq 0$, then $h_{out}(i) \geq 1$.

Appendix A2. Proof of Property 2

(1) Let the neighboring nodes directed to node A be 1, 2, $\cdots$, p. Because $d_{out}(j) = 1$, $j = 1, 2, \cdots, p$, so $h_{in}(A) = H(s_{1A} d_{out}(1), s_{2A} d_{out}(2), \cdots, s_{pA} d_{out}(p)) = H(s_{1A}, s_{2A}, \cdots, s_{pA}) = h_I(A)$. Similarly, the nodes directed from node A are denoted as 1, 2, $\cdots$,

q, then $h_{out}(B)=H(s_{B1}d_{in}(1), s_{B2}d_{in}(2), \cdots, s_{Bq}d_{in}(q))=H(s_{B1}, s_{B2}, \cdots, s_{Bq})=h_O(A)$. Here $h_I(A)$ and $h_O(A)$ represent node A's in-h-degree and out-h-degree, respectively.

(2) It is clear based on the definition of bi-directional h-index.

(3) By Property 1 and Property 2(1), Property 2(3) can be proved.

Appendix A3. Proof of Theorem 1

According to the definition of operator H, for any node i and any integer $n \geq 0$, we have $h_{in}^{(n)}(i) \geq 0$ and $h_{out}^{(n)}(i) \geq 0$, namely, sequences $\{h_{in}^{(n)}(i)\}_{n=0}^{\infty}$ and $\{h_{out}^{(n)}(i)\}_{n=0}^{\infty}$ all take zero as the lower bound. By definition of operator H, $h_{in}^{(0)}(i) = d_{in}(i) \geq h_{in}^{(1)}(i)$ and $h_{out}^{(0)}(i) = d_{out}(i) \geq h_{out}^{(1)}(i)$. Next, applying mathematical induction, we will prove that $h_{in}^{(n)}(i) \geq h_{in}^{(n+1)}(i)$ and $h_{out}^{(n)}(i) \geq h_{out}^{(n+1)}(i)$ for any node i and any integer $n \geq 0$. Suppose $h_{out}^{(m)}(i) \geq h_{out}^{(m+1)}(i)$ and $h_{in}^{(m)}(i) \geq h_{in}^{(m+1)}(i)$ for integer $n=m \geq 1$ and any node i, then by formulas (3) and (4),

$$\begin{aligned}
&h_{in}^{(m+2)}(i) = H(s_{k_1 i}h_{out}^{(m+1)}(k_1), s_{k_2 i}h_{out}^{(m+1)}(k_2), \cdots, s_{k_{d_{in}(i)}i}h_{out}^{(m+1)}(k_{d_{in}(i)}))\\
&\leq H(s_{k_1 i}h_{out}^{(m)}(k_1), s_{k_2 i}h_{out}^{(m)}(k_2), \cdots, s_{k_{d_{in}(i)}i}h_{out}^{(m)}(k_{d_{in}(i)}))\\
&= h_{in}^{(m+1)}(i)
\end{aligned}$$

and

$$\begin{aligned}
&h_{out}^{(m+2)}(i) = H(s_{ij_1}h_{in}^{(m+1)}(j_1), s_{ij_2}h_{in}^{(m+1)}(j_2), \cdots, s_{ij_{d_{out}(i)}}h_{in}^{(m+1)}(j_{d_{out}(i)}))\\
&\leq H(s_{ij_1}h_{in}^{(m)}(j_1), s_{ij_2}h_{in}^{(m)}(j_2), \cdots, s_{ij_{d_{out}(i)}}h_{in}^{(m)}(j_{d_{out}(i)}))\\
&= h_{out}^{(m+1)}(i).
\end{aligned}$$

Therefore, the sequences $h_{in}^{(0)}(i), h_{in}^{(1)}(i), h_{in}^{(2)}(i), \cdots$ and $h_{out}^{(0)}(i), h_{out}^{(1)}(i), h_{out}^{(2)}(i), \cdots$ are non-increasing, and each element is non-negative, so they have nonnegative limits, respectively. Since each element of the two sequences is a nonnegative integer, so the two sequences converge in finite steps, respectively.

Appendix A4. Proof of Theorem 2

For convenience, here we only prove the case that a node is randomly selected and its g value is updated each time; the proof of the other case is similar. We introduce the updating time step $t \geq 0$ and denote $g_{in}^{(0)}(i) = d_{in}(i)$ and $g_{out}^{(0)}(i) = d_{out}(i)$ for every node i. If at time step t ($t \geq 1$) node i is selected, then update the g value of node i by formulas (9) and (10),

$$g_{in}^{(t)}(i) = H(s_{k_1 i}g_{out}^{(t-1)}(k_1), s_{k_2 i}g_{out}^{(t-1)}(k_2), \cdots, s_{k_{d_{in}(i)}i}g_{out}^{(t-1)}(k_{d_{in}(i)})), \tag{A.1}$$

$$g_{out}^{(t)}(i) = H(s_{ij_1}g_{in}^{(t-1)}(j_1), s_{ij_2}g_{in}^{(t-1)}(j_2), \cdots, s_{ij_{d_{out}(i)}}g_{in}^{(t-1)}(j_{d_{out}(i)})), \tag{A.2}$$

else $g_{in}^{(t)}(i)$ and $g_{out}^{(t)}(i)$ are the most recently updated values.

(1) Proof of convergence of the asynchronous updating process

For any node i, $g_{in}^{(0)}(i) = d_{in}(i) \geq 0$ and $g_{out}^{(0)}(i) = d_{out}(i) \geq 0$. From the definition of operator H, we have $g_{in}^{(t)}(i) \geq 0$ and $g_{out}^{(t)}(i) \geq 0$ for $t \geq 1$. Therefore, for any node i, $\{g_{in}^{(t)}(i)\}_{t=0}^{\infty}$ and $\{g_{out}^{(t)}(i)\}_{t=0}^{\infty}$ are nonnegative sequences. For any node i, regardless of whether node i is selected, there are $g_{in}^{(0)}(i) \geq g_{in}^{(1)}(i)$ and $g_{out}^{(0)}(i) \geq g_{out}^{(1)}(i)$. In fact, if node i is selected, then by formulas (A.1) and (A.2) $g_{in}^{(0)}(i) = d_{in}(i) \geq g_{in}^{(1)}(i)$ and $g_{out}^{(0)}(i) = d_{out}(i) \geq g_{out}^{(1)}(i)$, else $g_{in}^{(0)}(i) = g_{in}^{(1)}(i)$ and $g_{out}^{(0)}(i) = g_{out}^{(1)}(i)$. Next, applying mathematical induction, we will prove that $g_{in}^{(t)}(i) \geq g_{in}^{(t+1)}(i)$ and $g_{out}^{(t)}(i) \geq g_{out}^{(t+1)}(i)$ for any node i and any integer $t \geq 0$. Suppose $g_{out}^{(m)}(i) \geq g_{out}^{(m+1)}(i)$ and $g_{in}^{(m)}(i) \geq g_{in}^{(m+1)}(i)$ for integer $t=m \geq 1$ and any node i, if node i is selected at time step $m+1$, then by formulas (A.1) and (A.2),

$$\begin{aligned}
&g_{in}^{(m+2)}(i) = H(s_{k_1 i}g_{out}^{(m+1)}(k_1), s_{k_2 i}g_{out}^{(m+1)}(k_2), \cdots, s_{k_{d_{in}(i)}i}g_{out}^{(m+1)}(k_{d_{in}(i)}))\\
&\leq H(s_{k_1 i}g_{out}^{(m)}(k_1), s_{k_2 i}g_{out}^{(m)}(k_2), \cdots, s_{k_{d_{in}(i)}i}g_{out}^{(m)}(k_{d_{in}(i)}))\\
&= g_{in}^{(m+1)}(i)
\end{aligned}$$

and

$$\begin{aligned} g_{out}^{(m+2)}(i) &= H(s_{ij_1}g_{in}^{(m+1)}(j_1), s_{ij_2}g_{in}^{(m+1)}(j_2), \cdots, s_{ij_{d_{out}(i)}}g_{in}^{(m+1)}(j_{d_{out}(i)})) \\ &\leq H(s_{ij_1}g_{in}^{(m)}(j_1), s_{ij_2}g_{in}^{(m)}(j_2), \cdots, s_{ij_{d_{out}(i)}}g_{in}^{(m)}(j_{d_{out}(i)})) \\ &= g_{out}^{(m+1)}(i). \end{aligned}$$

If node i is not updated at time step $m+1$, then $g_{out}^{(m+2)}(i) = g_{out}^{(m+1)}(i)$ and $g_{in}^{(m+2)}(i) = g_{in}^{(m+1)}(i)$. Therefore, $\{g_{in}^{(t)}(i)\}_{t=0}^{\infty}$ and $\{g_{out}^{(t)}(i)\}_{t=0}^{\infty}$ are all non-increasing for any node i.

Because $\{g_{in}^{(t)}(i)\}_{t=0}^{\infty}$ and $\{g_{out}^{(t)}(i)\}_{t=0}^{\infty}$ are two non-increasing and nonnegative sequences, so $\{g_{in}^{(t)}(i)\}_{t=0}^{\infty}$ and $\{g_{out}^{(t)}(i)\}_{t=0}^{\infty}$ have nonnegative limits, respectively. Because each element of the two sequences is a nonnegative integer, in finite steps the two sequences converge respectively. The limits are denoted as $g_{in}^{\infty}(i)$ and $g_{out}^{\infty}(i)$, respectively, that is $\lim_{t\to\infty} g_{in}^{(t)}(i) = g_{in}^{\infty}(i)$ and $\lim_{t\to\infty} g_{out}^{(t)}(i) = g_{out}^{\infty}(i)$.

(2) Proof that the asynchronous and synchronous updating processes have the same limits

First, we prove the following two lemmas.

Lemma 1. *For every node i and any given update time $t \geq 0$, there exist $t_1 \geq t$ and $t_1' \geq t$, such that $g_{in}^{(t_1)}(i) \geq h_{in}^{(t)}(i)$ and $g_{out}^{(t_1')}(i) \geq h_{out}^{(t)}(i)$, here t_1 and t_1'* increase *with the increasing of t.*

Proof. Applying mathematical induction, we can prove Lemma 1. When $t=0$, let $t_1 = t_1' = 0$, then $g_{in}^{(0)}(i) = d_{in}(i) \geq d_{in}(i) = h_{in}^{(0)}(i)$ and $g_{out}^{(0)}(i) = d_{out}(i) \geq d_{out}(i) = h_{out}^{(0)}(i)$ for every node i. Suppose that there exist $t_1 \geq T$ and $t_1' \geq T$ that make the lemma correct for every node i. Next we prove it to also be valid for $t=T+1$. If node i is selected when $t=T+1$, then let $t_1 = t_1' = T+1$, there are,

$$\begin{aligned} g_{in}^{(t_1)}(i) &= g_{in}^{(T+1)}(i) = H(s_{k_1 i}g_{out}^{(T)}(k_1), s_{k_2 i}g_{out}^{(T)}(k_2), \cdots, s_{k_{d_{in}(i)}i}g_{out}^{(T)}(k_{d_{in}(i)})) \\ &\geq H(s_{k_1 i}h_{out}^{(T)}(k_1), s_{k_2 i}h_{out}^{(T)}(k_2), \cdots, s_{k_{d_{in}(i)}i}h_{out}^{(T)}(k_{d_{in}(i)})) \\ &= h_{in}^{(T+1)}(i) = h_{in}^{(t)}(i) \end{aligned}$$

and

$$\begin{aligned} g_{out}^{(t_1')}(i) &= g_{out}^{(T+1)}(i) = H(s_{ij_1}g_{in}^{(T)}(j_1), s_{ij_2}g_{in}^{(T)}(j_2), \cdots, s_{ij_{d_{out}(i)}}g_{in}^{(T)}(j_{d_{out}(i)})) \\ &\geq H(s_{ij_1}h_{in}^{(T)}(j_1), s_{ij_2}h_{in}^{(T)}(j_2), \cdots, s_{ij_{d_{out}(i)}}h_{in}^{(T)}(j_{d_{out}(i)})) \\ &= h_{out}^{(T+1)}(i) = h_{out}^{(t)}(i). \end{aligned}$$

Otherwise, if node i is not selected when $t=T+1$, then, $g_{in}^{(t_1)}(i) = g_{in}^{(T+1)}(i) = g_{in}^{(T)}(i) \geq h_{in}^{(T)}(i) \geq h_{in}^{(T+1)}(i) = h_{in}^{(t)}(i)$ and $g_{out}^{(t_1')}(i) = g_{out}^{(T+1)}(i) = g_{out}^{(T)}(i) \geq h_{out}^{(T)}(i) \geq h_{out}^{(T+1)}(i) = h_{out}^{(t)}(i)$. Therefore, Lemma 1 was established when $t=T+1$. In summary, the proof of Lemma 1 is completed by mathematical induction.

Lemma 2. *For every node i and any given update time $t \geq 0$, there exist $t_2 \geq t$ and $t_2' \geq t$, such that $g_{in}^{(t_2)}(i) \leq h_{in}^{(t)}(i)$ and $g_{out}^{(t_2')}(i) \leq h_{out}^{(t)}(i)$, where t_2 and t_2' increase with the increasing of t.*

Proof. Applying mathematical induction, we can prove Lemma 2. When $t=0$, let $t_2 = t_2' = t = 0$, then $g_{in}^{(0)}(i) = d_{in}(i) \leq d_{in}(i) = h_{in}^{(0)}(i)$ and $g_{out}^{(0)}(i) = d_{out}(i) \leq d_{out}(i) = h_{out}^{(0)}(i)$. Suppose that when $t=T \geq 1$, there exist $t_2(i) \geq t$ and $t_2'(i) \geq t$ for every node i, such that $g_{in}^{(t_2)}(i) \leq h_{in}^{(T)}(i)$ and $g_{out}^{(t_2')}(i) \leq h_{out}^{(T)}(i)$. Next we prove it to also be valid for $t=T+1$. When $t=T+1$, from the definition of operator H and the property that $g_{in}^{(t)}(i)$ is non-increasing when t increases, we have,

$$\begin{aligned} h_{in}^{(T+1)}(i) &= H(s_{k_1 i}h_{out}^{(T)}(k_1), s_{k_2 i}h_{out}^{(T)}(k_2), \cdots, s_{k_{d_{in}(i)}i}h_{out}^{(T)}(k_{d_{in}(i)})) \\ &\geq H(s_{k_1 i}g_{out}^{(t_2'(k_1))}(k_1), s_{k_2 i}g_{out}^{(t_2'(k_2))}(k_2), \cdots, s_{k_{d_{in}(i)}i}g_{out}^{(t_2'(k_{d_{in}(i)}))}(k_{d_{in}(i)})) \\ &\geq H(s_{k_1 i}g_{out}^{(r)}(k_1), s_{k_2 i}g_{out}^{(r)}(k_2), \cdots, s_{k_{d_{in}(i)}i}g_{out}^{(r)}(k_{d_{in}(i)})) \\ &= g_{in}^{(r+1)}(i). \end{aligned}$$

Here $r = \max\{t_2^{'}(k_1), t_2^{'}(k_2), \cdots, t_2^{'}(k_{d_{in}(i)})\} \geq T$, so $r+1 \geq T+1$. Because $g_{in}^{(t)}(i)$ is non-increasing when t increases, therefore, when $t_2 \geq r+1$, $g_{in}^{(t_2)}(i) \leq h_{in}^{(T+1)}(i)$ and $t_2 \geq T+1$. So, for node i, we can select a time $t_2 \geq r+1$ and it is greater than the t_2 at time T, such that $g_{in}^{(t_2)}(i) \leq h_{in}^{(T+1)}(i)$.

In a similar way, from the definition of operator H and the property that $g_{out}^{(t)}(i)$ is non-increasing when t increases, we have,

$$\begin{aligned} h_{out}^{(T+1)}(i) &= H(s_{ij_1} h_{in}^{(T)}(j_1), s_{ij_2} h_{in}^{(T)}(j_2), \cdots, s_{ij_{d_{out}(i)}} h_{in}^{(T)}(j_{d_{out}(i)})) \\ &\geq H(s_{ij_1} g_{in}^{(t_2(j_1))}(j_1), s_{ij_2} g_{in}^{(t_2(j_2))}(j_2), \cdots, s_{ij_{d_{out}(i)}} g_{in}^{(t_2(j_{d_{out}(i)}))}(j_{d_{out}(i)})) \\ &\geq H(s_{ij_1} g_{in}^{(r')}(j_1), s_{ij_2} g_{in}^{(r')}(j_2), \cdots, s_{ij_{d_{out}(i)}} g_{in}^{(r')}(j_{d_{out}(i)})) \\ &= g_{out}^{(r'+1)}(i). \end{aligned}$$

Here $r' = \max\{t_2(k_1), t_2(k_2), \cdots, t_2(k_{d_{in}(i)})\} \geq T$, so $r'+1 \geq T+1$. Because $g_{out}^{(t)}(i)$ is non-increasing when t increases, therefore, when $t_2^{'} \geq r'+1$, $g_{out}^{(t_2^{'})}(i) \leq h_{out}^{(T+1)}(i)$ and $t_2^{'} \geq T+1$. So, for node i, we can select a time $t_2^{'} \geq r'+1$ and it is greater than the $t_2^{'}$ at time T, such that $g_{out}^{(t_2^{'})}(i) \leq h_{out}^{(T+1)}(i)$. Therefore, the conclusion was established when $t = T+1$. In summary, the proof of Lemma 2 is completed by mathematical induction.

Second, we prove the two updating processes have the same limits.

Because the sequence $\lim_{t\to\infty} g_{in}^{(t)}(i) = g_{in}^{\infty}(i)$, according to Lemma 1 and Lemma 2, the subsequences $\{g_{in}^{(t_1)}(i)\}$ and $\{g_{in}^{(t_2)}(i)\}$ exist, they all converge to $g_{in}^{\infty}(i)$. Since $\lim_{t\to\infty} h_{in}^{(t)}(i) = h_{in}^{\infty}(i)$, also $g_{in}^{(t_1)}(i) \geq h_{in}^{(t)}(i)$ and $g_{in}^{(t_2)}(i) \leq h_{in}^{(t)}(i)$, we have $g_{in}^{(\infty)}(i) \geq h_{in}^{(\infty)}(i)$ and $g_{in}^{(\infty)}(i) \leq h_{in}^{(\infty)}(i)$. Therefore, $g_{in}^{(\infty)}(i) = h_{in}^{(\infty)}(i)$ for every node i. Similarly, $g_{out}^{(\infty)}(i) = h_{out}^{(\infty)}(i)$ for every node i. The proof of Theorem 2 is completed.

Appendix A5. The list of 24 leading business journals

See Table A1.

Table A1
The list of 24 leading business journals.

No	Journal Name	Abbreviation
1	Academy of Management Journal	ACAD MANAGE J
2	Academy of Management Review	ACAD MANAGE REV
3	The Accounting Review	ACCOUNT REV
4	Administrative Science Quarterly	ADMIN SCI QUART
5	Information Systems Research	INFORM SYST RES
6	Informs Journal on Computing	INFORMS J COMPUT
7	Journal of Accounting and Economics	J ACCOUNTING ECONOMI
8	Journal of Accounting Research	J ACCOUNTING RES
9	Journal of Consumer Research	J CONSUM RES
10	Journal of Finance	J FINANC
11	Journal of Financial Economics	J FINANC ECON
12	Journal of International Business Studies	J INT BUS STUD
13	Journal of Marketing	J MARKETING
14	Journal of Marketing Research	J MARKETING RES
15	Journal of Operations Management	J OPER MANAG
16	Manufacturing and Service Operations Management	M&SOM-MANUF SERV OP
17	Management Science	MANAGE SCI
18	Marketing Science	MARKET SCI
19	MIS Quarterly	MIS QUART
20	Operations Research	OPER RES
21	Organization Science	ORGAN SCI
22	Production and Operations Management	PROD OPER MANAG
23	The Review of Financial Studies	REV FINANC STUD
24	Strategic Management Journal	STRATEGIC MANAGE J

Source: http://jindal.utdallas.edu/the-utd-top-100-business-school-research-rankings/journals
Note: Ascending order by abbreviation.

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