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A THEORY OF CONTINUOUS RATES AND APPLICATIONS TO THE THEORY OF GROWTH AND OBSOLESCENCE RATES

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Abstract—For functions f of a continuous variable t, we define the term "rate" (as, e.g., rate of growth or of obsolescence) as the exponential function of the derivative of the logarithm of this function (i.e., $e^{(\log f)'}$). This replaces discrete calculations, such as $f(t+1)/f(t)$, which is not so appropriate in this continuous context. We investigate this transformation (which is in fact the exponential function of the Fechner law), and show that it indeed has all properties that we can expect from a "rate" function. We then apply these findings to the results of three previous papers and again prove the main results in this continuous setting.

1. INTRODUCTION

The literature on growth or obsolescence (aging) problems comprises about 3,000 pieces of literature (mainly journal articles) nowadays. In this number we only counted the publications from 1970 onward! It is clear that these topics are in the center of the informetricians' interests.

Obsolescence is the phenomenon of "the reduced use of literature (on a certain topic) in time." The basis for (international) obsolescence studies is the citation habits of authors (in the form of reference lists), insofar as their citations are in order (there are many misuses or drawbacks of citations, e.g., unnecessary self-citations, geographical or linguistical barriers, political reasons, the "you scratch my back and I'll scratch yours" syndrome, multiple authorship (counting problem), implicit citations, author's name alterations or misspellings, and so on). In general, however, we can say that nowadays citation studies are well accepted in science policy studies, as well as in obsolescence studies, as long as we are working with massive pieces of literature. For a detailed description of the many pros and cons on these topics, we refer the reader to the review in Egghe and Rousseau (1990), pp. 203–227, where many references are also presented.

Amongst the many authors who deal with obsolescence we refer, for example, to Brookes (1970a) and Line (1970), both dealing with the obsolescence of periodical literature—in this connection the notion of "half-life" is important (i.e., the period in time in which the periodical receives (or gives) 50% of its citations). Indeed, one can study reference lists of publications, in which case we are doing a synchronous study of obsolescence; or we can study the use of a publication from the year it was published, in which case we are doing a diachronous study of obsolescence (cf. the work of Stinson, 1981; Stinson & Lancaster, 1987; Carter & Line, 1974). Obsolescence can also be studied "locally" as the decline of the use (e.g., the number of borrowings) of a book in a library. The main references for this topic are Burrell (1985,1986,1987). The obsolescence function (describing the number of citations over time) has also been studied by Egghe and Rao (1992a), where it was shown that a lognormal function fits best.

Growth is the phenomenon of "increase of number of publications (on a certain topic) in time." It is clear that this is very important to measure; growth means space (of library shelves and library space, of computer memory, and so on), and space means money. Growth also has a sociological aspect, since the growth of literature implies potential accessibility problems for the user of this literature. A few references on growth are Archibald and Line (1990), Burrell (1989), Line and Roberts (1976), and the recent work of Egghe and Rao (1992b), which will be used frequently in the sequel.

Studying growth or obsolescence problems is important for predicting future evolutions, but several papers study the influence of growth on obsolescence. This was studied recently in Egghe (1993), but much earlier by Line (1970), Brookes (1970b), and Gama De Queiroz and Lancaster (1979–1981).

This combination of growth and obsolescence was the basis for the present paper. We note that growth and obsolescence can be studied by the same mathematical techniques. This can be illustrated as follows.

Let f be a real function of the real variable $t \in \mathbf{R}^+$ (the positive real numbers, including 0). In practise, we might think of a growth function $f = g$, where

$$g(t) = \text{the number of publications at time } t$$

or of an obsolescence function $f = c$, where

$$c(t) = \text{the number of references to publications that are } t \text{ years old}$$

(this is an example of synchronous obsolescence; an analogous example can be given in the diachronous case).

Although these interpretations are interesting and important, we do not have to make this distinction for the development of our theory. In this way we present a "combined" growth-obsolescence theory and, later on, we will present growth and obsolescence applications.

With f as above one is often interested in the rate (such as the growth rate or the obsolescence rate (also called aging rate)) of f, as expressed by

$$\alpha(t) = \frac{f(t+1)}{f(t)} \tag{1}$$

for all $t \in \mathbf{R}^+$. This measures the rate of change of the function f over a time period of length 1. Of course, α is dependent on t, in general. The use of eqn (1) is perfectly natural for functions $f(t)$, where t can only take integer values. When using eqn (1) for functions of a continuous variable t, there is something unnatural in it, since there is no real unity (i.e., time block of length 1).

We can compare this with working with $f(t + 1) - f(t)$ without ever using the derivative:

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}. \tag{2}$$

In the sequel we will define a "continuous analogue" of eqn (1) and show that this measure is the natural one to be used in the continuous setting. We will prove that a constant rate appears if and only if f is an exponential function. Our measure is also related to the sensation law of Fechner (see, e.g., Gleitman, 1981).

Three previous papers (Egghe & Rao, 1992a,1992b, and Egghe, 1993) are reconsidered and the results are proved again in this continuous setting. First of all we show that, if f is an obsolescence function c, the rate has a stable minimum attained after the maximum of c (see Egghe & Rao, 1992a). Secondly, if f is a growth function g, we re-examine the growth rates in the continuous setting as we did in Egghe and Rao (1992b). Finally, the influence of growth on obsolescence is studied as in Egghe (1993) in two ways. Firstly we prove that, in the synchronous case, obsolescence increases with growth. Secondly we show the opposite effect in the diachronous case.

2. CONTINUOUS RATES

Let f be a positive real function on the domain $\mathbf{R}^+$. As an analogue of $f'(t)$, replacing $f(t+1) - f(t)$, we define the **the rate of f** as:

$$R(f)(t) = e^{(\log f)'(t)}. \tag{3}$$

for $t \in \mathbf{R}^+$. R could also be called the "proportional derivative" of f. Intuitively, this is the continuous analogue of $f(t+1)/f(t)$, since the natural logarithm is applied. Indeed, symbolically using the connection $\approx$ between the discrete and continuous approach: $f'(t) \approx f(t+1) - f(t)$, we find:

$$e^{(\log f)'(t)} \approx e^{\log f(t+1) - \log f(t)} = \frac{f(t+1)}{f(t)}.$$

We will check some properties of the transformation R. Of course

$$R(f)(t) = e^{\frac{f'(t)}{f(t)}} \tag{4}$$

for $t \in \mathbf{R}^+$. We can also denote $R(f)(t) = (R^0 f)(t)$. Here, $R^0 f$ is related to the celebrated sensation law of Fechner stating that the "sensation (S) is proportional to the logarithm of the stimulus (I)", in a formula (see, e.g., Gleitman, 1981):

$$S = \log I \tag{5}$$

(or another logarithmic function $\log_a$). Hence, the change in sensation is measured as:

$$S' = (\log I)' = \frac{I'}{I} \tag{6}$$

analogously to the exponent in eqn (3) or eqn (4).

Examples of this follow. If you carry a weight of, say 1 kg, you will feel very clearly the addition of 1 kg; if you carry a weight of 20 kg, the increase to 21 kg will not be felt that clearly. Adding 1,000 books to a library that contains only 5,000 books will be experienced quickly by the library users; if, however, the library already has 500,000 books, the addition of another 1,000 books will not be seen immediately.

This "relative sensitivity" is also what we want to measure for "rates." Up to an exponential function, we define rate as the relative change f'/f (i.e., the change of the sensation) of a function f (i.e., a stimulus). The addition of the exponential function in eqns (3) and (4) is necessary to go back to the "level" of the function f (after having taken the logarithm) (see also the concrete calculations further on).

We have the following preliminary property, showing that we are headed in the right direction.

Proposition 2.1

1. *If f is a constant function, then*

$$R(f)(t) = 1 \tag{7}$$

for all $t \in \mathbf{R}^+$ and

$$\int_{t'=t}^{t'=t+1} R(f)(t')\,dt' = \frac{f(t+1)}{f(t)}. \tag{8}$$

2. *If f is an exponential function (say $f(t) = ca^t$), then*

$$R(f)(t) = a \tag{9}$$

for all $t \in \mathbf{R}^+$ and

$$\int_{t'=t}^{t'=t+1} R(f)(t')\,dt' = \frac{f(t+1)}{f(t)}. \tag{8}$$

Proof. See the appendix.

This result proves that R has the good properties for the "classical" growth or obsolescence function—the exponential one (note that 1. is, in fact, contained in 2., by taking $a = 1$).

For a linear growth function $f(t) = t$ we find $R(f)(t) = e^{1/t}$, which is decreasing with t. This is logical—the larger t (i.e., $f(t)$), the smaller is the relative growth if the absolute growth remains constant. We have also the following proposition.

Proposition 2.2
Let f be as above:

1. *f increases strictly in $t \in \mathbf{R}^+$ iff $R(f)(t) > 1$.*
2. *f decreases strictly in $t \in \mathbf{R}^+$ iff $R(f)(t) < 1$.*
3. *f is stationary in $t \in \mathbf{R}^+$ iff $R(f)(t) = 1$.*

Proof. See the appendix.

The relevance of the above proposition is that R indeed expresses "rates" (compare with the exponential function $f(t) = ca^t$, where a replaces $R(f)$ in the above assertions). The following result gives necessary and sufficient conditions for the "variability" of the rate.

Proposition 2.3
Let f be as above:

1. *$R(f)$ increases strictly iff $ff' > (f')^2$,*
2. *$R(f)$ decreases strictly iff $ff'' < (f')^2$,*
3. *$R(f)$ is constant iff $ff'' = (f')^2$,*

iff $f(t) = c.a^5$ for all $t \in \mathbf{R}^+$ (for certain constants c and a).

Proof. See the appendix.

These results prove that $R(f)$ is a good rate function of f. f is a growth function if $R(f) > 1$, and an aging (obsolescence) function if $R(f) < 1$ (proposition 2.1).

Furthermore, the rate $R(f)$ (of growth or obsolescence) is constant iff f is an exponential function (proposition 2.3) $f(t) = ca^t$ and in this case $R(f)$ is the rate a. All these results show that R is a good rate function.

We note that using $\alpha'(t)$ or $(\log \alpha)'(t)$ instead of R (with α as in eqn (2)) does not give the results of the above propositions, showing that these are not the right notions we are looking for (for example: $\alpha'(t) = (\log \alpha)'(t) = 0$ for the exponential function $f(t) = ca^t$, showing that we do not have here the rate a).

From the theory so far developed, it is clear that growth and obsolescence are just two examples (or applications) of the above theory. We will now prove some more results for obsolescence, respectively, growth situations and for the influence of growth on obsolescence. We base ourselves on the three papers, Egghe and Rao (1992a, 1992b) and Egghe (1993).

3. APPLICATIONS TO OBSOLESCENCE AND GROWTH

3.1 *Obsolescence*

In Egghe and Rao (1992a), the following theorem was proved.

THEOREM 3.1

The obsolescence rate as a function of t has a minimum in a point $t^* > t_0$, where t_0 is the maximum of the obsolescence function c.

The interest in this result lies in the observation that the rate function is "retarded" by the obsolescence function. Intuitively this is clear, since rates have an "averaging" effect (for more on this, see Egghe & Rao, 1992a).

Here we supposed an obsolescence function as in Fig. 1. Note that initially there is a growth, a phenomenon that is always encountered. This is, of course, no obstacle for our study, since we have a global theory of rates, that is, for any function f we have defined $R(f)$, the rate of f. Obsolescence functions do indeed increase in the beginning; it is the period when the scientific community becomes aware of this publication. The use increases until a point where the "real" obsolescence starts.

As explained above, Theorem 3.1 was proved for the obsolescence function α, as in formula (1). In the Appendix, this result is proved for $R(f)$ with $f = c$ as above.

Also in Egghe and Rao (1992a), we proved (both theoretically and by statistical fitting of several practical examples) that the lognormal distribution fits the graph of c best:

$$f(t) = c(t) = \frac{1}{t\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\log t - \mu}{\sigma}\right)^2}. \tag{10}$$

Here we find, for the aging rate,

$$R(c)(t) = e^{-\frac{1}{t}\left(1 + \frac{\log t - \mu}{\sigma^2}\right)}. \tag{11}$$

Here $\lim_{t\to\infty} R(c)(t) = 1$. Furthermore, if we can show that there is a $t > 0$ such that $R(c)(t) < 1$, then, by (A3) (see the appendix), $R(c)$ must have a minimum. $R(c)(t) < 1$ if and only if

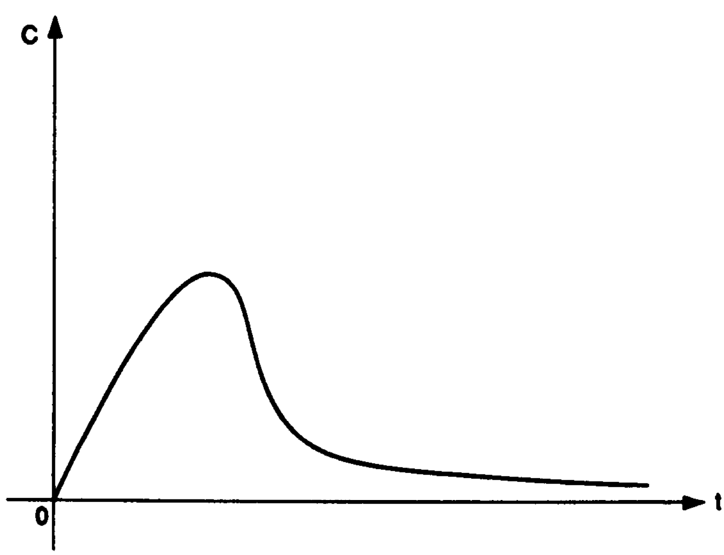

Fig. 1. Obsolescence function.

$$-\frac{1}{t}\left(1+\frac{\log t-\mu}{\sigma^2}\right)<0$$

iff

$$t>e^{\mu-\sigma^2}>0;$$

hence, such a t exists. Hence, for the lognormal function, $R(c)$ appears much as in Fig. 2 (cf. the analogy with the result in Egghe & Rao, 1992a). The minimum value is in $t^* = e^{-\mu+\sigma^2-1}$, as can easily be seen by calculating $R'(c)(t)$.

3.2 *Growth*

In Egghe and Rao (1992b), a classification of growth curves, based on growth rates, has been made. We will now investigate whether we can do the same analysis for our rate (growth rate here), function R. We use the same growth models as in Egghe and Rao (1992b).

3.2.1 *The exponential model.* This case is already known:

$$R(g)(t)=a$$

by proposition 2.1, for the growth function

$$g(t)=ca^t \quad (t\geq 0).$$

We again find the constant function, as in Fig. 3.

3.2.2 *The logistic model.* Here

$$g(t)=\frac{1}{k+ab^t} \tag{12}$$

where $k,a>0$, $0<b<1$, $t\geq 0$. Now:

$$R(g)(t)=b^{-\frac{ab^t}{k+ab^t}}. \tag{13}$$

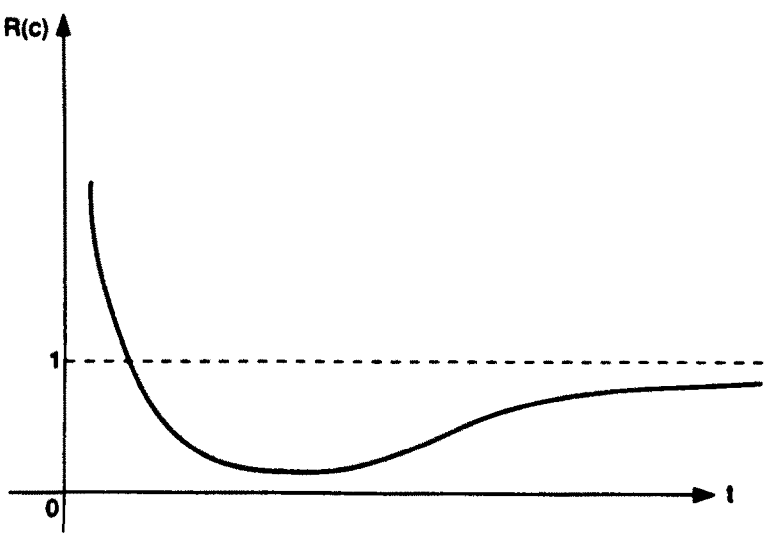

Fig. 2. $R(c)$ for the lognormal function.

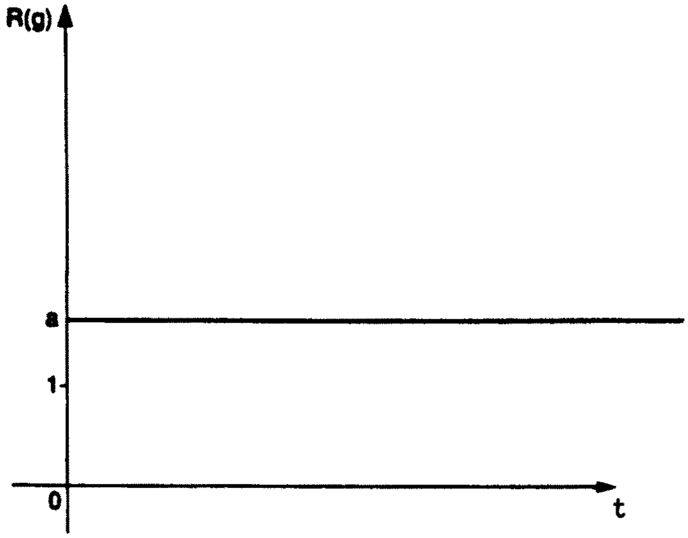

Fig. 3. $R(g)$ for the exponential model.

Now:

$$R(g)(0) = b^{-\frac{a}{k+a}} > 1, \lim_{t\to\infty} R(g)(t) = 1.$$

$R'(g)(t) < 0$ for all $t \geq 0$ and $R'(g)(0) > -\infty$.

We again find the graph as in Fig. 4 (see Egghe and Rao, 1992b).

3.2.3 *Gompertz model.* Here:

$$g(t) = DA^{B^t}, \tag{14}$$

where $\log A \log B > 0$, $D > 0$, $t \geq 0$. Now:

$$R(g)(t) = e^{B^t \log A \log B} \tag{15}$$

is positive everywhere and $R(g)(0) = e^{\log A \log B} > 1$.

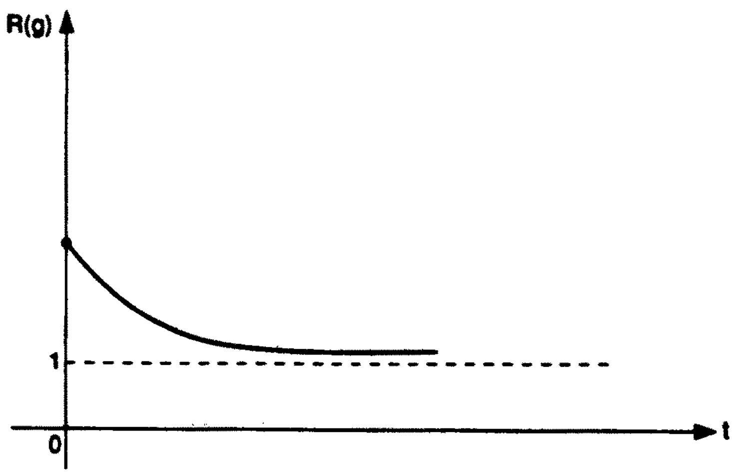

Fig. 4. $R(g)$ for the logistic model.

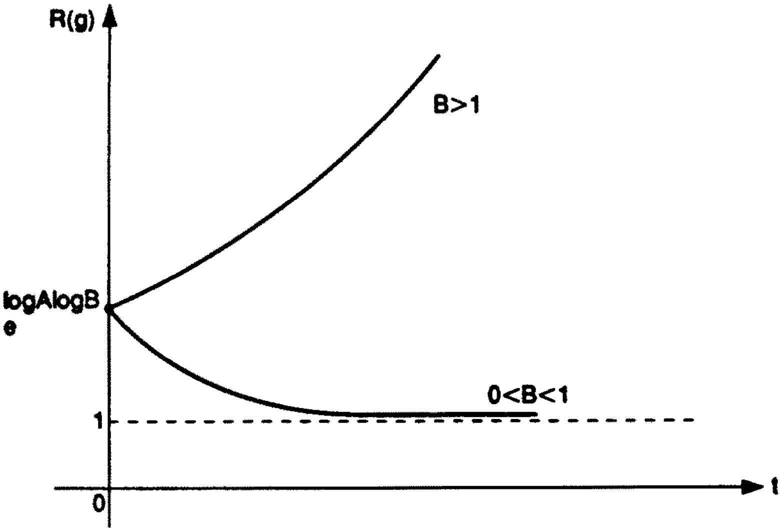

Fig. 5. $R(g)$ for Gompertz models.

R is increasing for $B > 1$ and decreasing for $0 < B < 1$. We refind both figures as in Egghe and Rao (1992b); see Fig. 5.

3.2.4 *Ware's model.* Now:

$$g(t) = \delta(1 - \varphi^{-t}) \tag{16}$$

with $\delta > 0$, $\varphi > 1$, $t \geq 0$. We have:

$$R(g)(t) = \varphi^{\frac{\varphi^{-t}}{1-\varphi^{-t}}}. \tag{17}$$

We have that $\lim_{t \underset{>}{\to} 0} R(g)(t) = +\infty$ and $\lim_{t \to \infty} R(g)(t) = 1$, and also that $R'(g)(t) < 0$ for all $t \geq 0$. We refind the result in Egghe and Rao (1992b); see Fig. 6.

3.2.5 *The power model.* The function g is now:

$$g(t) = \alpha + \beta t^{\gamma} \tag{18}$$

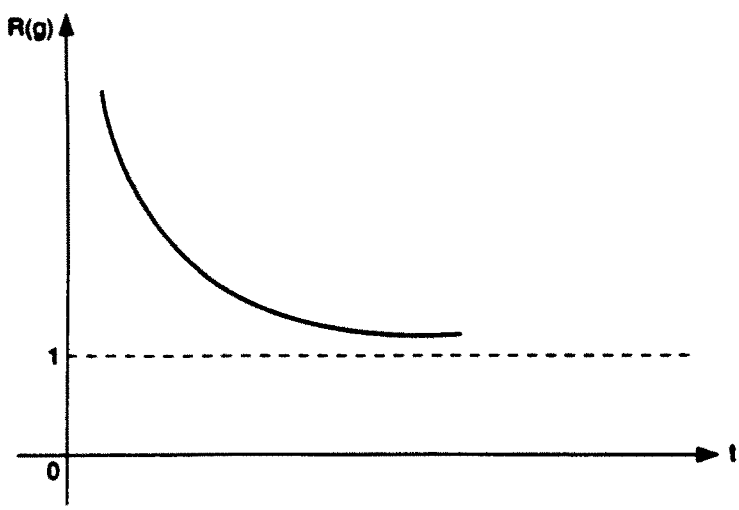

Fig. 6. $R(g)$ for Ware's model.

where $\gamma, \beta > 0$, $\alpha \geq 0$. Now:

$$R(g)(t) = e^{\gamma\beta \frac{t^{\gamma-1}}{\alpha + \beta t^{\gamma}}}. \tag{19}$$

$R(g)(0) = 1$ for $\gamma > 1$, $\alpha \neq 0$, $R(g)(0) = e^{\beta/\alpha} > 1$ for $\gamma = 1$ and $\lim_{t\to 0,>} R(g)(t) = +\infty$ for $0 < \gamma < 1$ or if $\alpha = 0$.

Always $\lim_{t\to\infty} R(g)(t) = 1$.

$$R'(g)(t) = \frac{R(g)(t)}{(\alpha + \beta t^{\gamma})^2} \gamma\beta t^{\gamma-2}(\alpha\gamma - \alpha - \beta t^{\gamma}) < 0,$$

except if $\gamma > 1$ and $t < (\alpha(\gamma - 1)/\beta)^{1/\gamma}$.

We again find the three figures as in Egghe and Rao (1992b), but with small changes in the parameters; see Figs. 7, 8, and 9.

3.3 *The influence of growth on obsolescence*

In Egghe (1992) we investigated the influence of growth on the obsolescence rate of the literature. We did this for both the synchronous and the diachronous obsolescence. In both cases we used $f(t + 1)/f(t)$ instead of $R(f)$. In this paper we will prove that the results obtained in Egghe (1992) are still true for rates expressed by $R(f)$.

We briefly repeat the methods that we introduced there and that we need here as well.

3.3.1 *Review of previous results.* In Egghe (1992), we supposed that the obsolescence in its pure form (i.e., per publication, hence not influenced by growth) can be described via a fixed exponential function of the type

$$c(t) = cb^t \tag{20}$$

(where $0 < b < 1$), whereas the growth function of the literature is described via another exponential function:

$$g(t) = ga^t \quad (a > 1). \tag{21}$$

In the synchronous case we studied a fixed time period $[0, T]$ in which the literature grows according to eqn (21), and where each piece of literature has a reference list for which

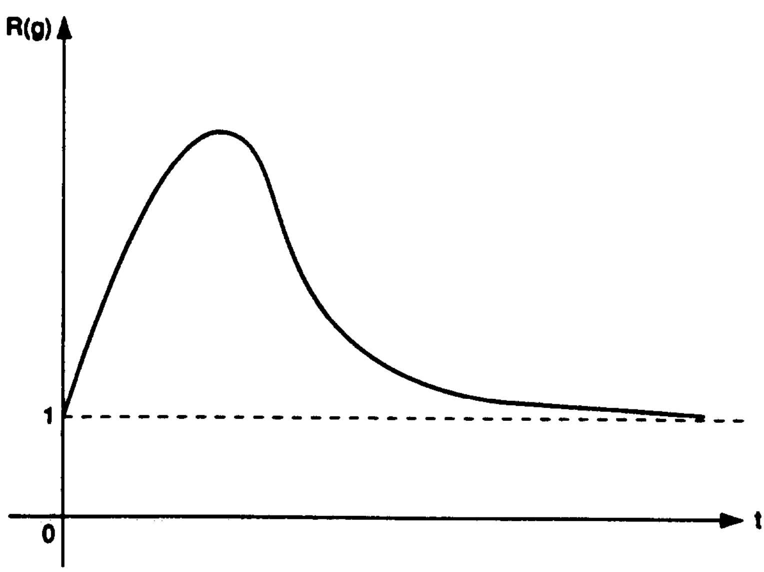

Fig. 7. $R(g)$ for the power model ($\gamma > 1$, $\alpha > 0$).

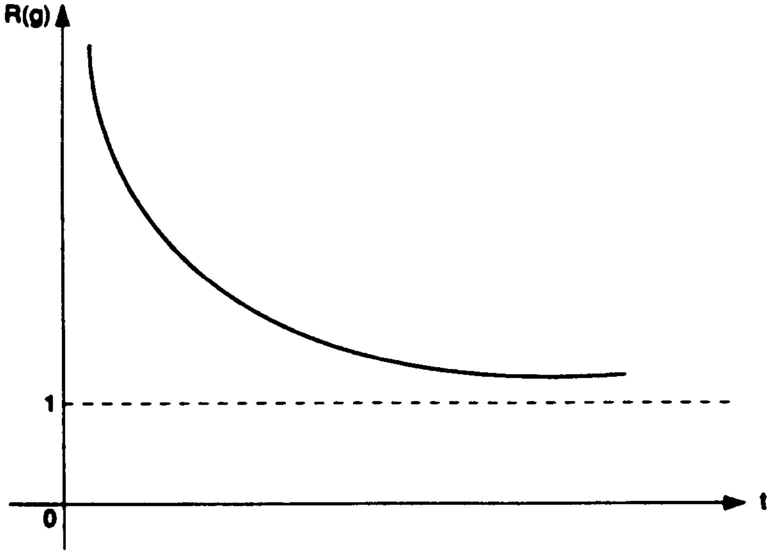

Fig. 8. $R(g)$ for the power model ($\alpha = 0$ or $0 < \gamma < 1$).

the age of the cited documents satisfies eqn (20). In this setting we found an *overall* obsolescence function, as in eqn (22):

$$\gamma(t) = \frac{gca^T}{\log b}\left(b^t - \frac{1}{a^t}\right) \quad (ab \neq 1). \tag{22}$$

Then we calculated $\gamma(t+1)/\gamma(t)$, and we could show that this function decreases with a; hence, the faster the growth, the higher the obsolescence. This result will be proved again, but for $R(\gamma)$, replacing $\gamma(t+1)/\gamma(t)$ (see the appendix).

In the diachronous case we have literature that grows in a time period $[0, T]$ and that is "followed" up to a time $T_0 > T$ (T_0 = the present) for possible citations. Again we supposed that growth is expressed via eqn (21), and that every piece of literature is cited $c(t)dt$

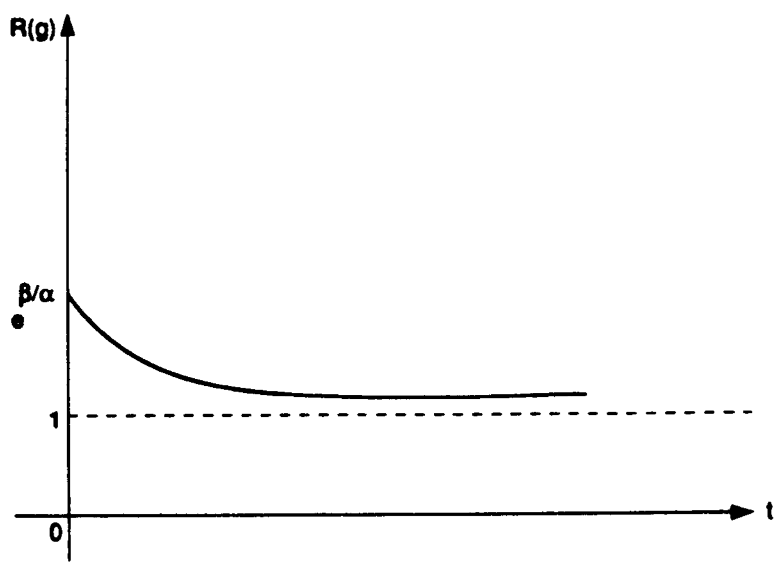

Fig. 9. $R(g)$ for the power model ($\gamma = 1$, $\alpha > 0$).

times (formula 20), t years after its publication, up to the time T_0. Here we found an overall obsolescence function as in eqn (23):

$$\gamma(t) = \begin{cases} gcb^t\left(\left(\dfrac{a}{b}\right)^t - 1\right)\dfrac{\log a}{\log a - \log b}, & t \le T \\ gcb^t\left(\left(\dfrac{a}{b}\right)^T - 1\right)\dfrac{\log a}{\log a - \log b}, & T \le t \le T_0. \end{cases} \quad (23)$$

We then proved for $\gamma(t+1)/\gamma(t)$ the opposite effect as in the synchronous case: We proved that the faster the growth, the smaller the obsolescence. This result will also be reproved, but for $R(\gamma)$ replacing $\gamma(t+1)/\gamma(t)$ (see the appendix).

4. CONCLUSION

The definition of "rate of a function f" as given by

$$R(f) = e^{(\log f)'} \quad (24)$$

proves to be the right formulation for expressing proportional changes in time for functions dependent on continuous variables. We therefore hope that, whenever the studied functions have continuous variables, rates will be expressed via eqn (24) and not via $f(t+1)/f(t)$, which can be used in the case where the variable t varies discretely (i.e., $t = 0,1,2,3,\ldots$). In general, the study of eqn (24) is easier than that of the discrete rate.

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APPENDIX

Proof of PROPOSITION 2.1.

1. If $f = c$, a constant, then it is clear that eqns (7) and (8) are satisfied.
2. If $f(t) = ca^t$, then $\log f(t) = \log c + t - \log a$. So $(\log f)'(t) = \log a$, and hence

$$R(f)(t) = e^{\log a} = a.$$

Hence also:

$$\int_{t'=t}^{t'=t+1} R(f)(t')\,dt' = a = \frac{f(t+1)}{f(t)} \qquad \square$$

Proof of PROPOSITION 2.2.

$$R(f)(t) = e^{(\log f)'(t)} > 1 \quad \text{iff } (\log f)'(t) > 0$$

iff $\log f$, hence f, strictly increases in t. The same argument can be given in the other cases. $\square$

Proof of PROPOSITION 2.3.

1. $R(f)$ increases strictly iff $(\log f)'(t)$ increases strictly iff

$$(\log f)''(t) > 0.$$

Now:

$$(\log f)'' = \frac{ff'' - (f')^2}{f^2}.$$

Hence, 1 follows.

2. Analogously.
3. $R(f)$ is constant iff $(\log f)'' = 0$ (this can be seen as above). The general solution of the above second-order equation is $(\log f)(t) = mt + n$, where m and n are constants. Hence:

$$f(t) = e^{mt+n}$$

$$f(t) = ca^t,$$

where $c = e^n$ and $a = e^m$ are constants. $\square$

Proof of THEOREM 3.1.

$$R(c)(t) = e^{(\log c)'(t)} = e^{\frac{c'(t)}{c(t)}}.$$

Hence:

$$R'(c)(t) = e^{\frac{c'(t)}{c(t)}} \cdot \frac{c(t)c''(t) - (c'(t))^2}{c(t)^2}$$

$$= R(c)(t) \cdot \frac{c(t)c''(t) - (c'(t))^2}{c(t)^2}. \qquad \text{(A1)}$$

Let t_0 be the point in which c attains its maximum and $t_1 > t_0$ the point of osculation of c.

For $t \in \,]0, t_0[$ we have $c'(t) > 0$ and $c''(t) < 0$. Consequently, $R'(c)(t) < 0$.

Let $t \in \,]t_0, t_1[$. Then $c'(t) < 0$ and $c''(t) < 0$. Hence again, $R'(c)(t) < 0$.

Let $t > t_1$. Then $c'(t) < 0$ and $c''(t) > 0$. Now there are two cases:

(1) $c(t)c''(t) \geq (c'(t))^2$. Then $R'(c)(t) \geq 0$.
(2) $c(t)c''(t) < (c'(t))^2$. Then $R'(c)(t) < 0$.

Since c' is continuous, we have in case (1) the existence of a point $t^* > t_0$ such that $R'(c)(t^*) = 0$. Hence in t^* the minimum of R is attained (see Fig. A1). In case (2), $R'(c) < 0$ all the time, while $R(c) \geq 0$. Hence $R(c)$ has a horizontal asymptote and $\lim_{t\to\infty} R(c)(t) \geq 0$. Here we can take $t^* = \infty$ (see Fig. A2). The point t^* (in case (1)) satisfies:

$$c'(t^*) = -\sqrt{c(t^*)c''(t^*)}. \tag{A2}$$

This follows from (A1) and the fact that $c'(t^*)$ is negative. □

Note that for c as in Fig. 1 also:

$$\lim_{t\to 0} R(c)(t) = \infty. \tag{A3}$$

Proof of the influence of growth on synchronous obsolescence

THEOREM 3.2
$R(\gamma)$ decreases with a. Hence, the faster the growth (a), the higher the obsolescence.

Proof. We have by eqn (22):

$$R(\gamma)(t) = e^{\left(\log\left(\frac{gca^T}{\log b}\left(b^t - \frac{1}{a^t}\right)\right)\right)'} = e^{\frac{b^t \log b + \frac{\log a}{a^t}}{b^t - \frac{1}{a^t}}} \tag{A4}$$

decreases with a iff

$$\frac{\partial R(\gamma)}{\partial a} < 0. \tag{A5}$$

This condition is equivalent with (after some calculation):

$$(ab)^t(1 - t\log(ab)) < 1. \tag{A6}$$

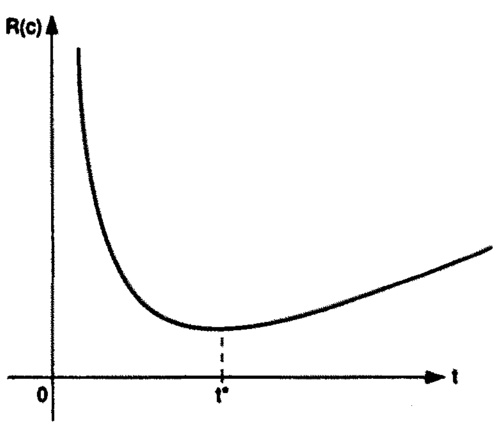

Fig. A1. Obsolescence rate, case (1).

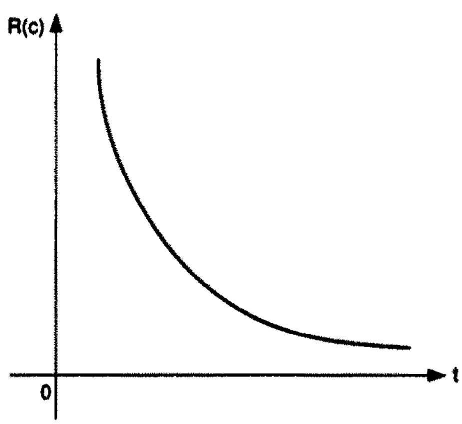

Fig. A2. Obsolescence rate, case (2).

But:

$$(ab)^{-t} = e^{-t\log(ab)}$$
$$> 1 - t\log(ab),$$

whence eqns (A6) and (A7). □

Proof of the influence of growth on diachronous obsolescence

THEOREM 3.3

$R(\gamma)$ increases with a. Hence, the faster the growth (a) the lower the obsolescence.

Proof. In this case we have, by eqn (23):

$$R(\gamma)(t) = \begin{cases} e^{\frac{\left(\frac{a}{b}\right)^t \log a - \log b}{\left(\frac{a}{b}\right)^t - 1}}, & t \leq T \\ b, \quad T \leq t \leq T_0. \end{cases} \tag{A7}$$

The latter case represents the period of non-growth, and hence it is natural to find an $R(\gamma)$ independent of a (the same result was found in the more intricate case of $\gamma(t+1)/\gamma(t)$ in Egghe (1992)). For $t \leq T$, we have that $R(\gamma)$ increases with a iff

$$\frac{\partial R(\gamma)}{\partial a} > 0. \tag{A8}$$

This condition is equivalent with (after some calculation):

$$\left(\frac{a}{b}\right)^t - 1 > t\log\left(\frac{a}{b}\right). \tag{A9}$$

But:

$$\left(\frac{a}{b}\right)^t = e^{t\log\left(\frac{a}{b}\right)} > 1 + t\log\left(\frac{a}{b}\right);$$

whence eqns (A8) and (A9). □