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Further characterizations of the Hirsch index

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Abstract The Hirsch index is a number that synthesizes a researcher's output. It is defined as the maximum number h such that the researcher has h papers with at least h citations each. Four characterizations of the Hirsch index are suggested. The most compact one relies on the interpretation of the index as providing the number of valuable papers in an output and postulates three axioms. One, only cited papers can be valuable. Two, the index is strongly monotonic: if output x has more papers than output y and each paper in x has more citations than the most cited paper in y, then x has more valuable papers than y. And three, the minimum amount of citations under which a paper becomes valuable is different for each paper.

Keywords Hirsch index · Axiomatic characterization · Publications · Citations · Scientific productivity index

Introduction

Let a researcher's output be characterized by a set of papers and, for each paper, the number of citations of that paper. The Hirsch (2005) index transforms a researcher's output into a number, interpreted as a measure of the output's scientific value or impact. Formally, the Hirsch index of an output x is the maximum number h of papers in x having at least h citations each. For instance, if x is the output consisting of one paper that is cited ten times and y is the output consisting of ten papers each of which is cited once, then x and y have both the same Hirsch index: 1.

Woeginger (2008a, b) and Quesada (2009, 2010) have already suggested characterizations of the Hirsch index, those by Woeginger (2008a, b) and Quesada (2009) hinging on the property of monotonicity: more papers or citations do not lower the index. Marchant

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(2009), instead of characterizing the index itself, characterizes the ranking that the Hirsch index induces on outputs. This paper suggests four characterizations of the Hirsch index.

The two formulated in Proposition 4.4 are essentially the same result, though the one in Proposition 4.4(ii) can be compared with the characterization of the number of papers index offered in Proposition 4.5(ii). From this comparison emerges the impression that the difference between the two indices can be explained in terms of what is the smallest output deserving one unit of the index.

The other two characterizations (Propositions 4.2, 4.3) are motivated by the presumption that the index associated with an output establishes the number of valuable papers in that output. Proposition 4.2 is based on just three axioms. First, the number of cited papers is an upper bound to the index, so being cited becomes a necessary condition for being valuable. Second, if output x has more papers than y and each paper in x has more citations than each paper in y, then x must have more valuable papers than y. And third, the number of citations necessary to make a paper valuable is different for each paper. This property is weaker than the plausible condition that, for $n > m$, the minimum number of citations making the nth most cited paper valuable is higher than the minimum number of citations making the mth most cited paper valuable [Quesada (2010) considers axioms bearing some resemblance to this requirement].

Proposition 4.3 is obtained from Proposition 4.2 by dispensing with the third axiom, which is replaced by the conjunction of independence of irrelevant citations, consistency and the condition that having a paper matching the number of citations of a valuable paper need not suffice to make that paper valuable. This condition and consistency rely on the presumption that if the nth most cited paper is valuable, then, for $m < n$, the mth most cited paper is also valuable. According to consistency, if the nth most cited paper in output x is not valuable and the nth most cited paper in output y has the same number of citations as the nth most cited paper in x, then the nth most cited paper in y is neither valuable. The independence condition asserts that more citations to papers already declared valuable do not increase the number of valuable papers.

Propositions 4.2 and 4.3 can at least be viewed as characterizations complementing those by Woeginger (2008a, b), whose axioms (at least, axioms A1 and D) seem to presume that the units of measurement of the index are citations rather than papers.

Definitions

Denote by $\mathbb{N}$ the set of non-negative integers. Define V to be the set of all vectors $x = (x_1, x_2, \ldots, x_n)$ such that: (i) $n \in \mathbb{N}\backslash\{0\}$; (ii) $x_1 \geq x_2 \geq \cdots \geq x_n$; and (iii) for all $i \in \{1, \ldots, n\}$, $x_i \in \mathbb{N}$. The member $x = (x_1, \ldots, x_n)$ of V represents a research output consisting of n papers such that, for each $i \in \{1, \ldots, n\}$, x_i is the number of citations obtained by the ith most cited paper in output x. The set $X = V \cup \{\varnothing\}$ of research outputs (or outputs, for short) is obtained from V by adding the empty output, represented by $\varnothing$. For $x \in X$, p_x is the number of papers in output x and c_x is the number of papers in x having some citation. Let $c^0 = \varnothing$ and, for $n \in \mathbb{N}\backslash\{0\}$, define c^n to be the output consisting of n papers with n citations each.

Definition 2.1 A research output index (or index, for short) is a mapping f: $X \rightarrow \mathbb{N}$.

An index deals with the problem of reducing any output $x \in X$ to a number $f(x)$ that measures some property of x: quality, impact, value.... Requiring $f(x)$ to belong to $\mathbb{N}$ is restrictive. This restriction is justified by the presumption that $f(x)$ provides the number of papers considered valuable in output x.

Definition 2.2 The Hirsch (2005) index is the index h such that, for all $x \in X$: (i) if $c_x = 0$, then $h(x) = 0$; and (ii) if $c_x \neq 0$, then $h(x) = \max\{n \in \{1, \ldots, c_x\}: x_n \geq n\}$.

Definition 2.3 The number of papers index is the index p such that, for all $x \in X$, $p(x) = p_x$.

Definition 2.4 The binary relation $\geq$ on V is such that, for all $x \in V$ and $y \in V$, $x \geq y$ if and only: (i) if $p_x \geq p_y$; and (ii) for all $i \in \{1, \ldots, p_y\}$, $x_i \geq y_i$. The binary relation $\geq$ is extended to X by letting, for all $x \in X$, $x \geq \varnothing$.

The interpretation of $x \geq y$ is that output x dominates output y, in the sense that x has at least as many papers as y and, for all $i \in \{1, \ldots, p_y\}$, the ith most cited paper in x has at least as many citations as the ith most cited paper in y.

Definition 2.5 The binary relation $\gg$ on V is such that, for all $x \in V$ and $y \in V$, $x \gg y$ if and only if: (i) $p_x > p_y$; and (ii) for all $i \in \{1, \ldots, p_x\}$, $x_i > y_1$. The binary relation $\gg$ is extended to X by defining, for all $x \in X$, $x \gg \varnothing$ if and only if $x \in V$.

Having $x \gg y$ means that output x has more papers than output y and, moreover, that each paper in x has more citations than the most cited paper in y. The binary relation $\gg$ represents a possible way of defining a strict dominance relation among outputs.

Definition 2.6 The binary relation $<_{\mathrm{L}}$ on X is defined as follows. On the one hand, for all $x \in X$, $\varnothing <_{\mathrm{L}} x$ if and only if $x \in V$. On the other hand, for all $x \in V$ and $y \in V$, look for the smallest $i \in \mathbb{N}\backslash\{0\}$ such that $x_i \neq y_i$. If there is no such i, then $x <_{\mathrm{L}} y$ if and only if $p_x < p_y$. Otherwise, $x <_{\mathrm{L}} y$ if and only if $x_i < y_i$.

The binary relation $<_{\mathrm{L}}$ is the lexicographic ordering: $x <_{\mathrm{L}} y$ if the smallest i such that $x_i \neq y_i$ satisfies $x_i < y_i$, with the empty set assumed to be smaller than any number. For instance, $\varnothing <_{\mathrm{L}} (0) <_{\mathrm{L}} (0, 0)$, $(2, 1) <_{\mathrm{L}} (2, 2)$, $(2, 2, 2) <_{\mathrm{L}} (3, 1)$, and $(1, 1) <_{\mathrm{L}} (1, 1, 0)$.

Axioms

The axioms presume that $f(x)$ is interpreted as the number of valuable papers in output x. More specifically, the intended meaning of $f(x) = n$ is that only the n most cited papers in x are valuable. Hence, $f(x) = 0$ means that there is no valuable paper in x and, for $n \geq 1$, $f(x) = n$ means that the set of valuable papers in $x = (x_1, \ldots, x_n, \ldots, x_{p_x})$ is given by $\{1, \ldots, n\}$. Accordingly, $f(x)$ defines an initial interval of valuable papers in a sequence of papers $(1, \ldots, p_x)$ listed from most to least cited. A continuity property is implicit in this interpretation: for paper $n + 1$ in the list to become valuable it is necessary that paper n be valuable. Also implicit is the idea that having more citations makes it more likely for a paper to become valuable.

PAP. *Papers as the units of measurement*. For all $x \in X$, $f(x) \leq p_x$.
CIT. *Cited papers as the units of measurement*. For all $x \in X$, $f(x) \leq c_x$.
SMO. *Strong monotonicity*. For all $x \in X$ and $y \in X$, $x \gg y$ implies $f(x) > f(y)$.

SMO states a sufficient condition for an output x to contain more valuable papers than another output y: that x has more papers than y and that each paper in x receives more citations than the most cited paper in y. SMO bears some similitude with axiom D in Woeginger (2008a, p. 227; b, p. 301), which holds that $f(x) > f(y)$ when x is obtained from y by adding a paper with $f(x)$ citations and next increasing the number of citations of each paper by at least one. A significant difference is that D appears to require that $f(x)$ be interpreted as a number of citations, whereas SMO does not force $f(x)$ to be necessarily measured in citations.

CVX. *Convexity*. For all $x \in X$, $y \in X$ and $z \in X$, if $x \geq z \geq y$ and $f(x) = f(y)$, then $f(z) = f(y)$.

CVX states that if, according to the dominance relation $\geq$, output z lies between two outputs x and y having the same index, then z must have the same index as x and y. CVX can also be viewed as a monotonicity condition: if the changes from y to $x \geq y$ do not modify the index, then smaller changes should also be incapable of modifying the index. CVX is weaker than Woeginger's (2008a, p. 225; b, p. 299) monotonicity.

MON. *Monotonicity*. For all $x \in X$ and $y \in X$, $x \geq y$ implies $f(x) \geq f(y)$.

If f satisfies MON, then $x \geq z \geq y$ implies $f(x) \geq f(z) \geq f(y)$. In view of this, $f(x) = f(y)$ implies $f(z) = f(y)$. This proves that MON implies CVX. The index f such that, for all $x \in X$, $f(x) = h(x)$ if $h \geq 2$, $f(x) = 1$ if $h(x) = 0$, and $f(x) = 0$ if $h(x) = 1$ satisfies CVX, but not MON.

Woeginger (2008a, p. 225; b, p. 299) makes an index (interpreted as a measure of scientific impact) satisfy MON by definition. But MON seems questionable when indices yield average or representative values rather than global or aggregate values. For instance, the index a that associates with output x the average number of citations, which appears to be a reasonable index, fails to satisfy MON: if x is the output consisting of 10 papers each of which has 100 citations and y is obtained from x by adding a paper with one citation, $y \geq x$, $a(x) = 100$ but $a(y) = 91$.

IIC. *Independence of irrelevant citations*. For all $x \in X$ such that $f(x) \geq 1$ and $k \in \{1, \ldots, f(x)\}$, if $y \in X$ is obtained from x by adding one citation to paper k, then $f(y) = f(x)$.

IIC seems to be a natural implication of the way $f(x)$ is interpreted: if $f(x) = n$ and more citations are added to any of the first n most cited papers, no additional paper could become valuable. By IIC, the number of valuable paper does not change when a valuable paper receives an additional citation.

CON. *Consistency*. For all $x \in X$, $y \in X$ and $n \in \mathbb{N}\backslash\{0, 1\}$, if $f(x) < n$ and $y_n = x_n$, then $f(y) < n$, where, for all $z \in X$, $z_n = \varnothing$ if $n > p_z$.

CON may also be seen as a natural implication of the way $f(x)$ is interpreted. For suppose that the nth most cited paper in output x is not valuable. This suggests that the number x_n of citations accumulated by that paper are not enough. Now, consider any other output y in which the nth most cited paper just gathers x_n citations. A principle of consistency seems to justify the view that, if x_n citations were not enough to make the nth most cited paper valuable in x, then that same amount of citations cannot make the nth most cited paper in y valuable. CON just expresses this principle.

Definition 3.1 The threshold function t_f associated with an index f is the mapping t_f: $\mathbb{N}\backslash\{0\} \rightarrow \mathbb{N} \cup \{\varnothing\}$ such that, for all $n \in \mathbb{N}\backslash\{0\}$: (i) if there is no $x \in X$ such that $f(x) = n$, then $t_f(n) = \varnothing$; and (ii) otherwise, $t_f(n)$ is the smallest member of $\mathbb{N}$ such that, for some $x \in X$, $x_n = t_f(n)$ and $f(x) = n$.

For index f, the threshold $t_f(n)$ of an nth most cited paper under index f is the number $t_f(n)$, if it exists, satisfying: (i) for some $x \in X$ such that $x_n = t_f(n)$, $f(x) = n$; and (ii) for no $x \in X$ such that $x_n < t_f(n)$, $f(x) = n$. In words, $t_f(n)$ is the minimum number of citations of an nth most cited paper making paper n valuable.

Remark 3.2 The threshold function t_h that corresponds to the Hirsch index is such that, for all $n \in \mathbb{N}\backslash\{0\}$, $t_h(n) = n$.

DIF. *Different thresholds*. For all $n \in \mathbb{N}\backslash\{0\}$ and $m \in \mathbb{N}\backslash\{0, n\}$ such that $t_f(n) \neq \varnothing \neq t_f(m)$, $t_f(n) \neq t_f(m)$.

Given the presumption that citations determine whether a paper is valuable, one may wonder what effort is considered necessary to make a paper valuable. DIF asserts that the minimum effort (number of citations) that, in some case, suffices to make a paper valuable is different for any two papers, if that effort exists. In particular, DIF asserts that, for $n \neq m$, the necessary effort to make the nth most cited paper valuable is different from the one necessary to make the mth most cited paper valuable. This captures the idea that the task necessary to make a paper valuable is specific to that paper. The following axiom formalizes a similar idea.

SAM. *Doing the same is not always enough*. For every $n \in \mathbb{N}\backslash\{0\}$, there are $x \in X$ and $y \in X\backslash\{x\}$ such that $f(x) = n$, $f(y) \leq n$ and y differs from x only in that $y_{n+1} = x_n$.

Suppose that $f(x) = n$, so the x_n citations received by the nth most cited paper have been considered enough to make that paper valuable. SAM holds that, for the next most cited paper to become valuable, it is not always sufficient to obtain that same amount x_n of citations. According to SAM, what worked for some paper to make it valuable need not always work for another paper. SAM, like DIF, is motivated by the idea of differential effort: replicating what made a certain paper valuable does not ensure that another paper will become valuable.

For index f and $n \in \mathbb{N}$, define the basic output s_f^n with n valuable papers to be the member of X, if it exists, such that $f(s_f^n) = n$ and, for all $x \in X$ satisfying $f(x) = n$, $s_f^n <_{\mathrm{L}} x$. Hence, s_f^n is the smallest output, according to the lexicographic ordering $<_{\mathrm{L}}$, having index n. For $x \in V$ and $k \in \mathbb{N}\backslash\{0\}$, define $k \cdot x$ to be the member y of X such that: (i) $p_y = k \cdot p_x$; (ii) for all $i \in \{1, \ldots, p_x\}$, $y_i = k \cdot x_i$; and (iii) for all $i \in \{p_x + 1, \ldots, p_y\}$, $y_1 = k \cdot x_{p_x}$. For $x = \varnothing$ and $k \in \mathbb{N}\backslash\{0\}$, define $k \cdot x = x$. Loosely speaking, $k \cdot x$ is the output obtained from x by rescaling x by k, that is, by multiplying papers and citations by the scale k.

RET. *Constant returns to scale for basic outputs*. For all $n \in \mathbb{N}\backslash\{0\}$ and $k \in \mathbb{N}\backslash\{0\}$, $k \cdot s_f^n = s_f^{nk}$.

RET holds that if the basic output with $n \neq 0$ valuable papers is rescaled by k, then the result is the basic output with $k \cdot n$ valuable papers. This suggests that basic outputs are subject to constant returns to scale: by enlarging a basic output k times, the index is also enlarged k times.

Remark 3.3 The Hirsch index satisfies PAP, CIT, SMO, CVX, IIC, CON, DIF, SAM and RET.

It should not be difficult to verify that the Hirsch index satisfies PAP, CIT, SMO, CVX (given that MON implies CVX) and IIC. DIF follows from Remark 3.2. The Hirsch index satisfies RET because $k \cdot c^n = c^{nk}$ and, for all $n \in \mathbb{N}\backslash\{0\}$, $s_h^n = c^n$. With respect to CON, if $h(x) < n$, then the nth most cited paper in x, if it exists, has fewer than n citations. As $y_n < n$ implies $h(y) < n$, the Hirsch index satisfies CON. Finally, as regards SAM, note that $h(c^n) = n = h(x)$, where x is obtained from c^n by adding a paper with n citations.

It may be worth pointing out that the Egghe (2006) index g satisfies neither IIC nor CON: $g(2, 1) = 1$ and $g(3, 1) = 2$. And even if the Egghe index is defined so that PAP holds, it also fails to satisfy DIF, because $g(1) = 1$ and $g(3, 1) = 2$ imply $t_g(1) = t_g(2) = 1$.

Results

Lemma 4.1 *If an index f satisfies CIT and SMO, then:*

(i) *for all $x \in X$, $h(x) = 0$ implies $f(x) = 0$;*
(ii) *for all $n \in \mathbb{N}$, $f(c^n) = n$; and*
(iii) *for all $x \in X$, $f(x) \geq h(x)$.*

Proof (i) Let $x \in X$ satisfy $h(x) = 0$. Therefore, $c_x = 0$. By CIT, $f(x) \leq 0$. By definition of index, $f(x) \geq 0$. As a result, $f(x) = 0$. (ii) By (i), $f(c^0) = 0$. Taking $f(c^0) = 0$ as the base case of an induction argument, choose $n \in \mathbb{N}\backslash\{0\}$ and suppose that, for all $k \in \{0, \ldots, n - 1\}$, $f(c^k) = k$. In particular, $f(c^{n-1}) = n - 1$. Since $c^n \gg c^{n-1}$, by SMO, $f(c^n) > f(c^{n-1}) = n - 1$. This and $f(c^n) \in \mathbb{N}$ imply $f(c^n) \geq n$. By CIT, $f(c^n) \leq n$. As a consequence, $f(c^n) = n$. (iii) Let $x \in X$. If $h(x) = 0$, then, by (i), $f(x) \geq h(x)$. If $h(x) = n \geq 1$, then $x_n \geq n$. Hence, $x \gg c^{n-1}$. By SMO and (ii), $f(x) > f(c^{n-1}) = n - 1$. That is, $f(x) \geq n$. □

Proposition 4.2 *An index f satisfies CIT, SMO and DIF if and only if f is the Hirsch index.*

Proof "⇐" Remark 3.3. "⇒" Part 1: for all $n \in \mathbb{N}\backslash\{0\}$, $t_f(n) = n$. Since $c^1 = (1)$, by Lemma 4.1(ii), $f(1) = 1$. By Lemma 4.1(i), $f(0) = 0$. Accordingly, $t_f(1) = 1$. Taking $t_f(1) = 1$ as the base case of an induction argument, choose $n \in \mathbb{N}\backslash\{0, 1\}$ and suppose that, for all $k \in \{1, \ldots, n - 1\}$, $t_f(k) = k$. By Lemma 4.1(ii), for some $x \in X, f(x) = n$. This means that $t_f(n) \neq \emptyset$. By DIF and the induction hypothesis, $t_f(n) \notin \{t_f(1), \ldots, t_f(n-1)\} = \{1, \ldots, n-1\}$. Hence, $t_f(n) \geq n$. By Lemma 4.1(ii), $f(c^n) = n$. In view of this, $t_f(n) \leq n$ and, consequently, $t_f(n) = n$. Part 2: $f = h$. Choose $x \in X$. By Lemma 4.1(i), if $h(x) = 0$, then $f(x) = h(x)$. If $h(x) = n \geq 1$, then $p_x \geq n$ and $x_n \geq n$. By Lemma 4.1(iii), it is enough to show that $f(x) \leq n$. Case 1: $p_x = n$. As $x_n \geq n$, by CIT, $f(x) \leq n$. Case 2: $p_x > n$. Since $h(x) = n$, $x_{n+1} \leq n$. Given this, $f(x) = r > n$ would imply $t_f(r) \leq n$, contradicting part 1. □

Since DIF is probably the most questionable axiom in Proposition 4.2, Proposition 4.3 next provides another characterization in which the conjunction of IIC, CON and SAM replaces DIF.

Proposition 4.3 *An index f satisfies CIT, SMO, IIC, CON and SAM if and only if f is the Hirsch index.*

Proof "⇐" Remark 3.3. "⇒" Taking Lemma 4.1(i) as the base case of an induction argument, choose $n \geq 1$ and assume that, for all $k \in \{0, \ldots, n - 1\}$ and $x \in X$, $h(x) = k$ implies $f(x) = k$. Let $x \in X$ satisfy $h(x) = n$, so $p_x \geq n$ and $x_n \geq n$. By Lemma 4.1(iii), $f(x) \geq n$. It then remains to be shown that $f(x) \leq n$. Case 1: $p_x = n$. By CIT, $p_x = n$ and $x_n \geq n$ imply $f(x) \leq n$. Case 2: $p_x > n$. By SAM, there are $v \in X$ and $w \in X$ such that $f(v) = n$, $f(w) \leq n$ and $w_{n+1} = v_n$. If $v_n > n$, then $w_{n+1} = v_n$ implies $w \gg c^n$. By SMO and Lemma 4.1(ii), $f(w) > f(c^n) = n$: contradiction. Accordingly, $v_n \leq n$. If $v_n < n$, then $h(v) < n$ and, by the induction hypothesis, $f(v) < n$: contradiction. Summarizing, $v_n = n$. Therefore, $w_{n+1} = n$ and $f(w) \leq n$. Given that $h(x) = n$, $x_{n+1} \leq n$. Case 2a: $x_{n+1} = n$. Since $f(w) < n + 1$ and $x_{n+1} = w_{n+1}$, by CON, $f(x) < n + 1$. That is, $f(x) \leq n$. Case 2b: $x_{n+1} < n$. To prove that $f(x) \leq n$, suppose not: $f(x) > n$. Let $y \in X$ be obtained from x by replacing x_{n+1} with n (y belongs to X because $h(x) = n$ implies $x_n \geq n$). By IIC, $f(x) > n$ implies $f(y) = f(x) > n$. But, by case 2a, $y_{n+1} = n$ yields $f(y) = n$: contradiction. □

Despite the fact that it is probably harder to justify RET than to justify either DIF or SAM, the axiomatization of the Hirsch index invoking RET (Proposition 4.4 next) may be useful because it can be compared with a similar axiomatization of the number of papers index p (Proposition 4.5). In fact, since p satisfies IIC, Propositions 4.4(ii) and 4.5(ii) give the somewhat surprising impression that the difference between p and h can be traced back to the decision of whether the smallest output deserving index 1 is (0) or (1).

Proposition 4.4 An index f is the Hirsch index if and only if f satisfies:

(i) *CIT*, *RET*, *CVX*, *IIC* and $f(1) = 1$; *or*
(ii) *PAP*, *RET*, *CVX*, *IIC* and $s_f^1 = (1)$.

Proof "⇐" By Remark 3.3, the Hirsch index satisfies PAP, CIT, RET, CVX and IIC. Obviously, $h(1) = 1$. And it is not difficult to verify that the Hirsch index satisfies $s_h^1 = (1)$. "⇒" Part 1: for all $n \in \mathbb{N}\backslash\{0\}, f(c^n) = n$ and $s_f^n = c^n$. If (i) holds, by CIT, for all $x \in X$ such that $c_x = 0, f(x) = 0$. Consequently, $f(1) = 1$ means that $s_f^1 = (1)$. This and RET imply that, for all $n \in \mathbb{N}\backslash\{0\}, f(c^n) = n$ and, in addition, that, for all $n \in \mathbb{N}\backslash\{0\}$, $s_f^n = c^n$. If (ii) holds, then $s_f^1 = (1)$ and RET yield the same conclusion. Part 2: for all $x \in X$, $h(x) = 0$ implies $f(x) = 0$. Let $x \in X$ satisfy $h(x) = 0$, so $c_x = 0$. By definition of index, $f(x) \geq 0$. If $f(x) = n > 0$, then, since $x <_{\mathrm{L}} c^n$, $s_f^n \neq c^n$, which contradicts part 1.

Part 3: $f = h$. Letting $h(x) = n$, by part 2, $f(x) = n$ if $n = 0$. So let $n \geq 1$. Define $y \in X$ to be such that, for all $i \in \{1, \ldots, n\}$, $y_i = n$ and, for all $i \in \{n + 1, \ldots, p_x\}$, $y_i = x_i$. To prove that $f(y) = n$, suppose not: $f(y) = k \neq n$. If $k < n$, then, by part 1, $f(c^k) = k$ and $f(c^n) = n$. By CVX, $y \geq c^n \geq c^k$ and $f(y) = f(c^k) = k$ imply $f(c^n) = f(c^k)$: contradiction. If $k > n$, by part 1, $f(c^k) = k$ and $s_f^k = c^k$. Since $y <_{\mathrm{L}} c^k, f(y) = k$ would contradict $s_f^k = c^k$. The final conclusion is then that $f(y) = n$. Given this, by IIC, $f(x) = f(y) = n$. □

Proposition 4.5 *An index f is the number of papers index if and only if f satisfies*:

(i) *PAP*, *RET*, *CVX and* $f(0) = 1$; *or*
(ii) *PAP*, *RET*, *CVX and* $s_f^1 = (0)$.

Proof Let $(0^0) = \varnothing$ and, for $n \in \mathbb{N}\backslash\{0\}$, define (0^n) to be the member of X consisting of n papers without citations. "⇐" It is evident that $p(0) = 1$ and that p satisfies PAP. It is not difficult to verify that, for all $n \in \mathbb{N}$, $s_p^n = (0^n)$. Since $0^n = n{\cdot}(0)$, p satisfies RET. With respect to CVX, let $x \geq z \geq y$ and $p(x) = p(y)$. Hence, $p_x = p_y$. It then follows from $x \geq z$ that $p_x \geq p_y$ and from $z \geq y$ that $p_z \geq p_y$. As a result, $p_z = p_y$ and, consequently, $p(z) = p(y)$.

"⇒" Part 1: $f(\varnothing) = 0 = p(\varnothing)$. This follows from PAP and the fact that $f(\varnothing) \in \mathbb{N}$. Part 2: for all $n \in \mathbb{N}\backslash\{0\}, f(0^n) = p(0^n)$. If (i) holds, then, given that $\varnothing$ is the only member x of X such that $x <_{\mathrm{L}} (0), f(\varnothing) = 0$ and $f(0) = 1$ imply $(0) = s_f^1$. If (ii) holds, then $s_f^1 = (0)$ as well. By RET, for all $n \in \mathbb{N}\backslash\{0\}$, $s_f^n = n{\cdot}(0)$. Since $n{\cdot}(0) = (0^n)$, $s_f^n = (0^n)$. This means that, for all $n \in \mathbb{N}\backslash\{0\}, f(0^n) = n = p(0^n)$. Part 3: for all $x \in X$ such that $\varnothing \neq x \neq (0, \ldots, 0), f(x) = p(x)$. Suppose not: $f(x) = r \neq p(x)$. By PAP, $r < p_x$. Case 1: $r = 0$. By part 1, $f(\varnothing) = 0$. This, CVX and $x \geq (0) \geq \varnothing$ imply $f(0) = f(\varnothing) = 0$, which contradicts $f(0) = 1$. Case 2: $r \geq 1$. Let $n = p_x$. By part 2, $f(0^n) = n > r = f(0^r)$. By CVX, $f(x) = f(0^r) = r$ and $x \geq 0^n \geq 0^r$ imply $f(0^n) = f(0^r) = r$: contradiction. □

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