

Is there a need for fuzzy logic? ☆

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Abstract

“Is there a need for fuzzy logic?” is an issue which is associated with a long history of spirited discussions and debate. There are many misconceptions about fuzzy logic. Fuzzy logic is not fuzzy. Basically, fuzzy logic is a precise logic of imprecision and approximate reasoning. More specifically, fuzzy logic may be viewed as an attempt at formalization/mechanization of two remarkable human capabilities. First, the capability to converse, reason and make rational decisions in an environment of imprecision, uncertainty, incompleteness of information, conflicting information, partiality of truth and partiality of possibility – in short, in an environment of imperfect information. And second, the capability to perform a wide variety of physical and mental tasks without any measurements and any computations [L.A. Zadeh, From computing with numbers to computing with words – from manipulation of measurements to manipulation of perceptions, *IEEE Transactions on Circuits and Systems* 45 (1999) 105–119; L.A. Zadeh, A new direction in AI – toward a computational theory of perceptions, *AI Magazine* 22 (1) (2001) 73–84]. In fact, one of the principal contributions of fuzzy logic – a contribution which is widely unrecognized – is its high power of precisiation.

Fuzzy logic is much more than a logical system. It has many facets. The principal facets are: logical, fuzzy-set-theoretic, epistemic and relational. Most of the practical applications of fuzzy logic are associated with its relational facet.

In this paper, fuzzy logic is viewed in a nonstandard perspective. In this perspective, the cornerstones of fuzzy logic – and its principal distinguishing features – are: graduation, granulation, precisiation and the concept of a generalized constraint.

A concept which has a position of centrality in the nontraditional view of fuzzy logic is that of precisiation. Informally, precisiation is an operation which transforms an object, p , into an object, p^* , which in some specified sense is defined more precisely than p . The object of precisiation and the result of precisiation are referred to as *precisiend* and *precisiand*, respectively. In fuzzy logic, a differentiation is made between two meanings of precision – precision of value, *v-precision*, and precision of meaning, *m-precision*. Furthermore, in the case of *m-precision* a differentiation is made between *mh-precision*, which is human-oriented (nonmathematical), and *mm-precision*, which is machine-oriented (mathematical). A dictionary definition is a form of *mh-precision*, with the *definiens* and *definiendum* playing the roles of *precisiend* and *precisiand*, respectively. *Cointension* is a qualitative measure of the proximity of meanings of the *precisiend* and *precisiand*. A *precisiand* is *cointensive* if its meaning is close to the meaning of the *precisiend*.

A concept which plays a key role in the nontraditional view of fuzzy logic is that of a generalized constraint. If X is a variable then a generalized constraint on X , $GC(X)$, is expressed as $X \text{ isr } R$, where R is the constraining relation and r is an

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indexical variable which defines the modality of the constraint, that is, its semantics. The primary constraints are: possibilistic ($r = \text{blank}$), probabilistic ($r = p$) and veristic ($r = v$). The standard constraints are: bivalent possibilistic, probabilistic and bivalent veristic. In large measure, science is based on standard constraints.

Generalized constraints may be combined, qualified, projected, propagated and counterpropagated. The set of all generalized constraints, together with the rules which govern generation of generalized constraints, is referred to as the generalized constraint language, GCL. The standard constraint language, SCL, is a subset of GCL.

In fuzzy logic, propositions, predicates and other semantic entities are precisiated through translation into GCL. Equivalently, a semantic entity, p , may be precisiated by representing its meaning as a generalized constraint.

By construction, fuzzy logic has a much higher level of generality than bivalent logic. It is the generality of fuzzy logic that underlies much of what fuzzy logic has to offer. Among the important contributions of fuzzy logic are the following:

1. FL-generalization. Any bivalent-logic-based theory, T , may be FL-generalized, and hence upgraded, through addition to T of concepts and techniques drawn from fuzzy logic. Examples: fuzzy control, fuzzy linear programming, fuzzy probability theory and fuzzy topology.
2. Linguistic variables and fuzzy if–then rules. The formalism of linguistic variables and fuzzy if–then rules is, in effect, a powerful modeling language which is widely used in applications of fuzzy logic. Basically, the formalism serves as a means of summarization and information compression through the use of granulation.
3. Cointensive precisiation. Fuzzy logic has a high power of cointensive precisiation. This power is needed for a formulation of cointensive definitions of scientific concepts and cointensive formalization of human-centric fields such as economics, linguistics, law, conflict resolution, psychology and medicine.
4. NL-Computation (computing with words). Fuzzy logic serves as a basis for NL-Computation, that is, computation with information described in natural language. NL-Computation is of direct relevance to mechanization of natural language understanding and computation with imprecise probabilities. More generally, NL-Computation is needed for dealing with second-order uncertainty, that is, uncertainty about uncertainty, or uncertainty² for short.

In summary, progression from bivalent logic to fuzzy logic is a significant positive step in the evolution of science. In large measure, the real-world is a fuzzy world. To deal with fuzzy reality what is needed is fuzzy logic. In coming years, fuzzy logic is likely to grow in visibility, importance and acceptance.

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1. Introduction

Note. This paper is not polemical in nature nor does it dwell on techniques of fuzzy logic – such as the formalism of linguistic variables and the machinery of fuzzy if–then rules – which are well known and widely used. Rather, its principal objective is to draw attention to those unique features of fuzzy logic which are widely unrecognized – features which serve to explain why fuzzy logic is likely to grow significantly in visibility, importance and acceptance in coming years. The paper is in two parts. Part 1 may be viewed as an answer to the question: What is fuzzy logic? In Part 1, fuzzy logic is viewed in a nontraditional perspective. Part 2 attempts to clarify what fuzzy logic has to offer. For the reader's convenience, a synopsis of what fuzzy logic has to offer is included in Part 2, preceding Section 6.

“Is there a need for fuzzy logic?” is a question which is associated with a long history of spirited discussion and debate going back to the publication of my first paper on fuzzy sets in 1965 [74]. During much of its early history, fuzzy logic was for the most part an object of skepticism and derision, in part because the word “fuzzy” is generally used in a pejorative sense. Here are a few examples of what was said about fuzzy logic.

Professor Rudolf Kalman, a brilliant scientist and a friend, commenting in 1972 on my first exposition of the concept of a linguistic variable.

I would like to comment briefly on Professor Zadeh's presentation. His proposals could be severely, ferociously, even brutally criticized from a technical point of view. This would be out of place here. But a blunt question remains: Is Professor Zadeh presenting important ideas or is he indulging in wishful

thinking? No doubt Professor Zadeh's enthusiasm for fuzziness has been reinforced by the prevailing climate in the US – one of unprecedented permissiveness. “Fuzzification” is a kind of scientific permissiveness; it tends to result in socially appealing slogans unaccompanied by the discipline of hard scientific work and patient observation.

Let me say quite categorically that there is no such thing as a fuzzy concept, . . . We do talk about fuzzy things but they are not scientific concepts. Some people in the past have discovered certain interesting things, formulated their findings in a non-fuzzy way, and therefore we have progressed in science.

Professor William Kahan, a brilliant computer scientist, a colleague and a good friend, commenting on fuzzy logic.

“Fuzzy theory is wrong, wrong, and pernicious.” says William Kahan, a professor of computer sciences and mathematics at Cal whose Evans Hall office is a few doors from Zadeh's. “I cannot think of any problem that could not be solved better by ordinary logic.” “What Zadeh is saying is the same sort of things ‘Technology got us into this mess and now it can't get us out.’” Kahan says. “Well, technology did not get us into this mess. Greed and weakness and ambivalence got us into this mess. What we need is more logical thinking, not less. The danger of fuzzy theory is that it will encourage the sort of imprecise thinking that has brought us so much trouble.”

Professor Dennis Lindley, an eminent Bayesian, commenting on the adequacy of probability theory:

The only satisfactory description of uncertainty is probability. By this I mean that every uncertainty statement must be in the form of a probability; that several uncertainties must be combined using the rules of probability; and that the calculus of probabilities is adequate to handle all situations involving uncertainty. . . probability is the only sensible description of uncertainty and is adequate for all problems involving uncertainty. All other methods are inadequate. . . anything that can be done with fuzzy logic, belief functions, upper and lower probabilities, or any other alternative to probability can better be done with probability.

Professor Susan Haack, a prominent logician and philosopher, commenting on the need for fuzzy logic in her book “Deviant Logic Fuzzy Logic” [26]. “Since neither of the main arguments that are offered in its favor is acceptable, I conclude that we do not need fuzzy logic.”

What can be said about such views is that they reflect perceptions of fuzzy logic which are far removed from reality [18]. Furthermore, the critics did not recognize the basic importance of the concept of a linguistic variable – a concept which is unique to fuzzy logic [79]. The concept of a linguistic variable, in association with the calculi of fuzzy if-then rules, has a position of centrality in almost all applications of fuzzy logic [3,96,17,43,53,63].

Today, close to four decades after its conception, fuzzy logic is far less controversial than it was in the past. The wide-ranging impact of fuzzy logic is much too obvious to be ignored. A significant metric of the impact of fuzzy logic is the number of papers in the literature with “fuzzy” in the title. There are over 53,000 papers listed in the INSPEC database, and over 15,000 in the Math Science Net database. Another significant metric is the number of fuzzy-logic-related patents: over 4800 in Japan and over 1500 in the United States.

2. Part 1 What is fuzzy logic – a nontraditional view

2.1. Fuzzy logic – principal concepts and ideas

There are many misconceptions about fuzzy logic. To begin with, fuzzy logic is not fuzzy. Basically, fuzzy logic is a precise logic of imprecision and approximate reasoning [83,88]. More specifically, fuzzy logic may be viewed as an attempt at formalization/mechanization of two remarkable human capabilities. First, the capability to converse, reason and make rational decisions in an environment of imprecision, uncertainty, incompleteness of information, conflicting information, partiality of truth and partiality of possibility – in short, in an environment of imperfect information. And second, the capability to perform a wide variety of physical and mental tasks without any measurements and any computations [101]. Paradoxically, one of the principal

contributions of fuzzy logic – a contribution which is widely unrecognized – is its high power of precisiation of what is imprecise. This capability of fuzzy logic suggests, as was noted earlier, that it may find important applications in the realms of economics, linguistics, law and other human-centric fields.

Another source of confusion is the duality of meaning of fuzzy logic. In a narrow sense, fuzzy logic is a logical system which is a generalization of multivalued logic [27]. In a wide sense, which is in dominant use today, fuzzy logic, FL, is much more than a logical system. FL has many facets. The principal facets are: the logical facet, FLl; the fuzzy-set-theoretic facet, FLs; the epistemic facet, FLe; and the relational facet, FLr (Fig. 1).

The logical facet of FL, FLl, is fuzzy logic in its narrow sense. FLl may be viewed as a generalization of multivalued logic. The agenda of FLl is similar in spirit to the agenda of classical logic [24,27,56,47,19,49].

The fuzzy-set-theoretic facet, FLs, is focused on fuzzy sets, that is, on classes whose boundaries are unsharp. The theory of fuzzy sets is central to fuzzy logic. Historically, the theory of fuzzy sets preceded fuzzy logic in its wide sense.

The epistemic facet of FL, FLe, is concerned with knowledge representation, semantics of natural languages and information analysis. In FLe, a natural language is viewed as a system for describing perceptions. An important branch of FLe is possibility theory [85,86,89,15]. Another important branch of FLe is the computational theory of perceptions [99–101].

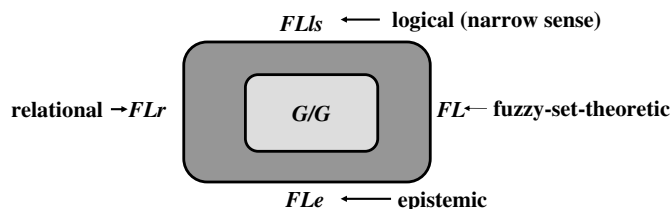
The relational facet, FLr, is focused on fuzzy relations and, more generally, on fuzzy dependencies. The concept of a linguistic variable – and the associated calculi of fuzzy if-then rules – play pivotal roles in almost all applications of fuzzy logic [7,68,12,20,34,3,28,70,13,17,30,37,71,8,72,43,39,57,1,53].

The basic concepts of graduation and granulation form the core of FL and are the principal distinguishing features of fuzzy logic. More specifically, in fuzzy logic everything is or is allowed to be graduated, that is, be a matter of degree or, equivalently, fuzzy.

Furthermore, in fuzzy logic everything is or is allowed to be granulated, with a granule being a clump of attribute-values drawn together by indistinguishability, similarity, proximity or functionality. Graduated granulation, or equivalently fuzzy granulation, is a unique feature of fuzzy logic. Graduated granulation is inspired by the way in which humans deal with complexity and imprecision [87,97,98].

An instance of granulation is the concept of a linguistic variable – a concept which was introduced in my 1973 paper “Outline of a new approach to the analysis of complex systems and decision processes” [79]. A simple example of a linguistic variable is shown in Fig. 2. Today, the concept of a linguistic variable is used in almost all applications of fuzzy logic, especially in the realms of control and consumer products. Granulation may be viewed as a form of information compression of variables and input/output relations.

An important concept which is related to the concept of a linguistic variable is the concept of a granular value [107]. More specifically, consider a variable, X , which takes values in U . Let u be a value of X . Informally, if u is known precisely, then u is referred to as a singular (point) value of X . If X is not known precisely, but there is some information which constrains possible values of u , then the constraint on u defines a granular value of X , (Fig. 3). For example, if what is known about u is that it is contained in an interval $[a, b]$, then $[a, b]$ is a granular value of X . A granular variable is a variable which takes granular values. In this sense, a linguistic variable is a granular variable which carries linguistic labels. It should be noted that a granular value of Age is not restricted to young, middle-aged or old. For example, “not very young” is an admissible granular value of



G/G: Graduation/Granulation

Fig. 1. Principal facets of fuzzy logic (FL). The core of FL is graduation/granulation, G/G.

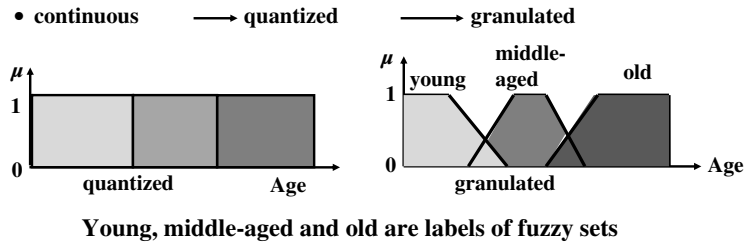


Fig. 2. Granulation of Age; young, middle-aged and old are linguistic (granular) values of age.

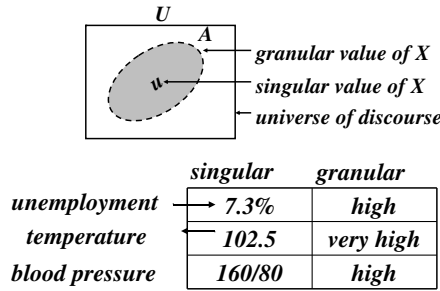


Fig. 3. Singular and granular values.

age [79,81]. As will be seen in Section 4, a granular value is defined by a generalized constraint. It is important to note that in fuzzy logic, in moving from numerical to linguistic variables, we are moving in a countertraditional direction. What my critics did not realize is that in moving in the countertraditional direction, we are sacrificing precision to achieve significant advantages down the line. This important feature of fuzzy logic is referred to as “the fuzzy logic gambit”. The fuzzy logic gambit will be discussed in greater detail in Section 8.

An early application of granulation to finite-state systems was described in my 1965 paper “Fuzzy sets and systems” [75]. Granulation plays a pivotal role in fuzzy control [77,78,42,20,34,28,30,71,72,22].

Granulation may be applied to arbitrarily complex objects. When convenient, the result of granulation may be referred to as the granuland. Application of granulation to an expression involves replacement of one or more singular variables in the expression with granular variables. For example, if

$$Z = X + Y$$

is an arithmetic expression, then its granuland may be expressed as

$$*Z = *X + *Y$$

in which starred variables are granular variables. In this sense, interval arithmetic may be viewed as the result of granulation of numerical arithmetic. Fig. 4 shows an application of granulation to a function, f .

The result of granulation, $*f$, is the fuzzy graph of f [80,84]. The fuzzy graph of f may be described as a collection of fuzzy if–then rules; it may be viewed as a summary of f . Describing a function as a collection of fuzzy if–then rules may be regarded as a form of information compression.

Application of granulation to probability distributions is illustrated in Fig. 5. The granules of X play the role of fuzzy events and the P_i are their granular probabilities [105].

It should be pointed out that in the case of probability distributions it is necessary to differentiate between granular probability distributions and granule-valued probability distributions [102,107] (Fig. 6). It should be noted that there is a connection between the concept of a granular probability distribution and the notion of Perfilieva transform [54].

An instance of a granule-valued distribution is a random set. There is a close connection between granule-valued distributions and the Dempster–Shafer theory of evidence [11,60,59]. More about granular and granule-valued distributions will be said in Section 4.

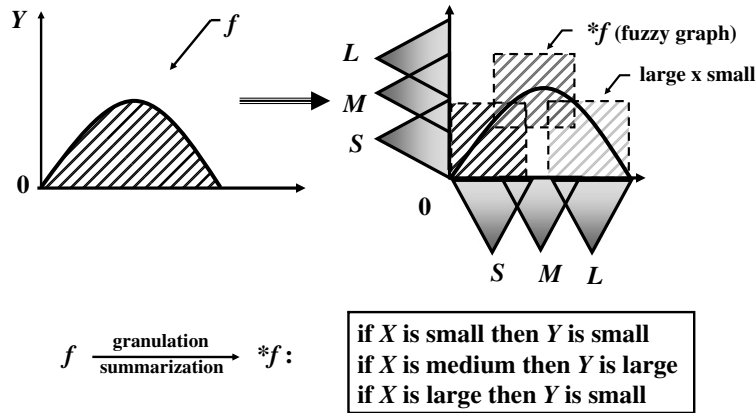


Fig. 4. Granulation of a function. S (small), M (medium) and L (large) are fuzzy sets. The granulation of f , $*f$, may be viewed as a summary of f .

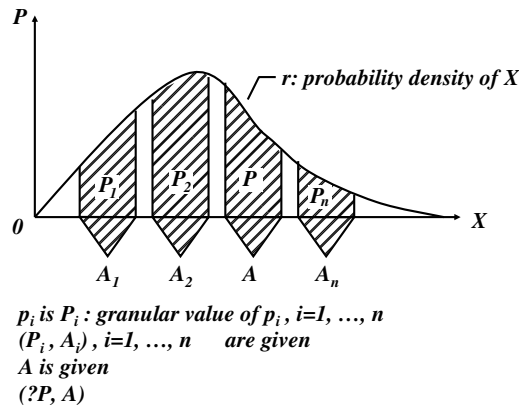


Fig. 5. Granulation and interpolation of a probability distribution.

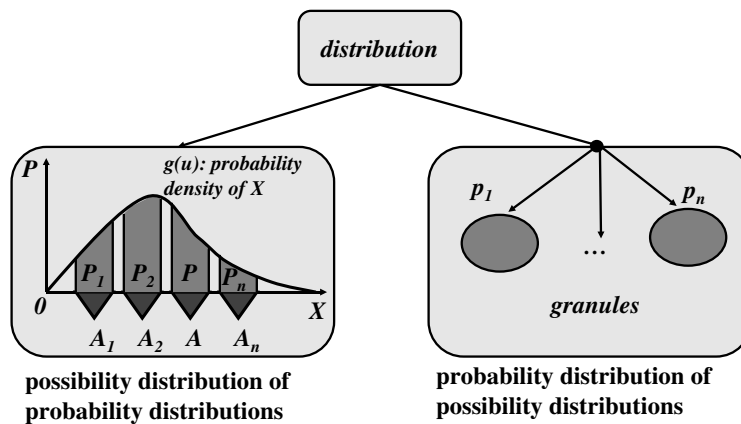


Fig. 6. Granular vs. granule-valued distributions.

A bit of history. The concept of granulation is implicit in my 1973 paper “Outline of a new approach to the analysis of complex systems and decision processes” [79] – a paper in which the basic concept of a linguistic variable was introduced. It was made explicit in my 1979 paper “Fuzzy sets and information granularity” [87]

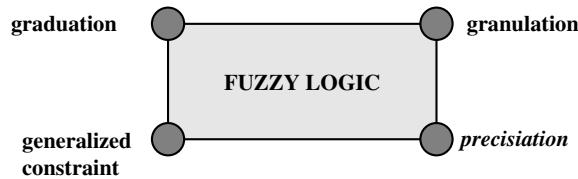


Fig. 7. The cornerstones of a nontraditional view of fuzzy logic.

and was developed further in my 1997 paper “Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic” [97]. This paper provided a basis for granular computing (GrC) – a mode of computation which is likely to grow in visibility and importance in coming years [98]. The term “granular computing” was suggested by Lin [40]. An important first book on granular computing was authored by Bargiela and Pedrycz [2]. There is a basic connection between granular computing and rough set theory [31].

In addition to the concepts of graduation and granulation there are two basic concepts which play important roles in the nontraditional view of fuzzy logic. These are the concepts of precision [93,103] and generalized constraint [95,99]. These concepts along with the concepts of graduation and granulation form the cornerstones of the nontraditional view of fuzzy logic (Fig. 7). The concepts of precision and cointensive precision are discussed in the following section. The concept of a generalized constraint is discussed in Section 4.

Note. In the sequel, when a comparison is made between bivalent logic and fuzzy logic, it should be understood that the objects of comparison are, on one side, bivalent logic and bivalent-logic-based probability theory, and on the other side, fuzzy logic and fuzzy-logic-based probability theory.

3. The concepts of precision and cointensive precision

There are not many concepts in science that are as pervasive as the concept of precision. There is an enormous literature. And yet, there are some important facets of the concept of precision which have received little if any attention. In particular, an issue that appears to have been overlooked relates to the need for differentiation between two forms of precision: precision of value, v-precision, and precision of meaning, m-precision [107]. Specifically, consider a variable X , whose value is not known precisely. In this event, the proposition $a \leq X \leq b$, where a and b are precisely specified numbers, is v-imprecise and m-precise. Similarly, the proposition: X is a normally distributed random variable with mean a and variance b , is v-imprecise and m-precise. On the other hand, the proposition: X is small, is both v-imprecise and m-imprecise if small is not defined precisely. If small is defined precisely as a fuzzy set, then the proposition in question is v-imprecise and m-precise.

Informally, precision is an operation which transforms an object, p , into another object, p^* , which is more precisely defined, in some specified sense, than p . The reverse applies to imprecision. In the realm of our discourse p is usually a proposition, predicate, question, command or, more generally, a linguistic expression which has a semantic identity. Unless stated to the contrary, p will be assumed to be a proposition. For convenience, the object and the result of precision are referred to as precisiend and precisiand, respectively (Fig. 8).

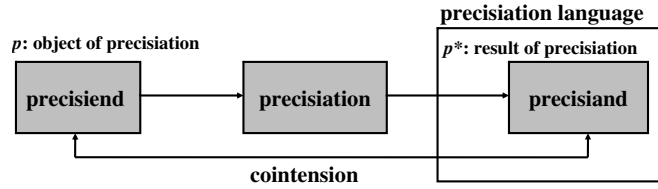
As in the case of precision/imprecision, there is a need for differentiation between v-precision/imprecision and m-precision/imprecision. Example:

$$X = 5 \begin{matrix} \xrightarrow{\text{v-impreciation}} \\ \xrightarrow{\text{m-impreciation}} \end{matrix} X \text{ is small}$$

$$X \text{ is small} \xrightarrow{\text{m-precision}} X \text{ is small (small is defined as a fuzzy set)}$$

It should be noted that data compression and summarization are forms of v-impreciation.

In the case of m-precision, there is a need for additional differentiation between m-precision which is human-oriented (nonmathematical), or mh-precision for short, and m-precision which is machine-ori-



precisiand = model of meaning
cointension = measure of closeness of meanings of p and p^*
 A **precisiand** may have many **precisiands**.

precision = translation into precision language

Fig. 8. Basic concepts relating to precision and cointension.

ented (mathematical), or mm-precision for short (Fig. 9). In this sense, a dictionary definition is a form of mh-precision, with the definiens and the definiendum playing the roles of precisiend and precisiand, respectively. A mathematical definition of a concept, say stability, is an instance of mm-precision.

A convenient illustration of mh-precision and mm-precision is provided by the concept of “bear market”.

mh-precisiand: declining stock market with expectation of further decline

mm-precisiand: 30% decline after 50 days, or a 13% decline after 145 days (Robert Shuster).

It is of interest to note that disambiguation is a form of m-precision. As an illustration, the alternative interpretations of the predicate, P : Most tall Swedes, are shown in Fig. 10.

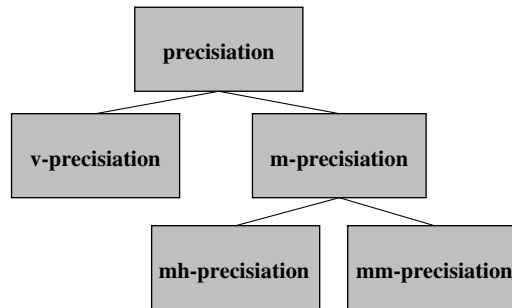


Fig. 9. Modalities of precision. mh-precision = nonmathematical precision of meaning; mm-precision = mathematical precision of meaning.

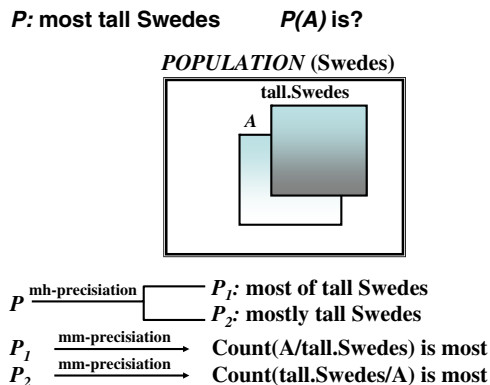


Fig. 10. Disambiguation of P : Most tall Swedes.

So far as v-impredication is concerned, there is a need for differentiation between (a) v-impredication which is forced; and (b) v-impredication which is deliberate. To illustrate, if the value of X is described imprecisely because the precise value of X is not known, then what is involved is forced v-impredication. On the other hand, if the value of X is described imprecisely even though the precise value of X is known, then v-impredication is deliberate.

What is the point of deliberate v-impredication? The rationale is that in many cases precision carries a cost. In such cases, deliberate v-impredication serves a useful purpose if it provides a way of reducing cost. Familiar examples of deliberate v-impredication are data compression and summarization. As will be seen at a later point, deliberate v-impredication underlies the fuzzy logic gambit (Section 8). The fuzzy logic gambit plays an important role in many applications of fuzzy logic, especially in the realm of consumer products – a realm in which cost is an important parameter.

A precisand may be precisiated in a large, perhaps unbounded, number of ways. As an illustration, Figs. 11a and 11b show some of the simpler mm-precisians of the predicate “approximately a ,” or $*a$ for short. Note that the simplest mm-precisand is a . This very simple mode of mm-precisiation is widely employed in many fields of science. Probability theory is a case in point. Most real-world probabilities are based on perceptions rather than on measurements. Perceptions are intrinsically imprecise. As a consequence, so are most

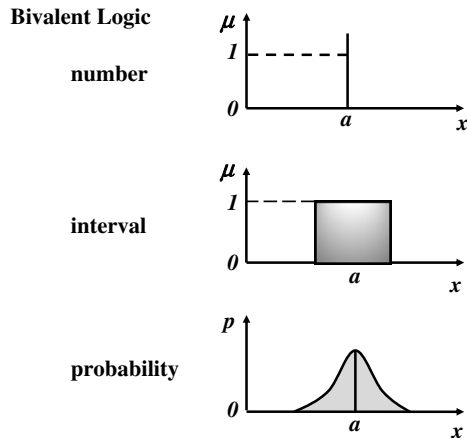


Fig. 11a. Alternative modes of mm-precisiation of “approximately a ,” $*a$, within the framework of bivalent logic.

Fuzzy Logic: Bivalent Logic + ...

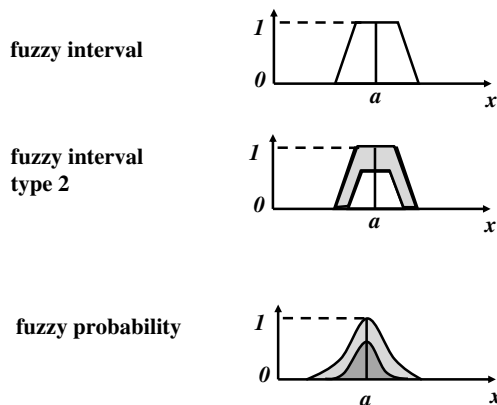


Fig. 11b. Alternative modes of mm-precisiation of “approximately a ,” $*a$, within the framework of fuzzy logic.

real-world probabilities. And yet, in most computations involving probabilities, probabilities are treated as exact numbers. For example, $^*0.7$ is treated as 0.7000.

Let p^* be a precisand of p . It is expedient to associate with p^* two basic metrics: (a) cointension – a qualitative measure of the proximity of the meanings of p and p^* ; and (b) the computational complexity of p^* . In general, cointension and computational complexity are covariant in the sense that an increase in the cointension of p^* is associated with an increase in the computational complexity of p^* .

Cointension is a new term which is in need of clarification. In logic, intension and extension are defined, respectively, as attribute-based meaning, or i-meaning for short, and name-based meaning, or e-meaning for short [38,10,5]. For our purposes, it is helpful to interpret attribute-based as measurement-based and, name-based as perception-based. What this means is that a precisand, p , is viewed as a perception of a concept, while its mm-precisand, p^* , is viewed as a measurement-based definition of p . For example, in the case of the concept of bear market, we have

mm-precisand: 30% decline after 50 days, or a 13% decline after 145 days (Robert Shuster).

More concretely, if p^* is an mm-precisand of p , then the cointension of p^* in relation to p , $C(p^*, p)$, is a qualitative measure of the degree of proximity of the i-meanings of p^* and p , or the proximity of the i-meaning of p^* and the e-meaning of p . p^* is cointensive if the degree of proximity is high. In the case of the bear market example, cointension is a measure of the degree to which the mm-precisand fits our perception of “bear market.” Although this degree cannot be assessed precisely, it is evident that the degree is not high.

Note. In a one-way communication via natural language between a human (sender) and a machine (recipient), mm-precisiation is a necessity because a machine cannot understand unprecisiated natural language. Two cases have to be considered. (a) Precisiation is carried out by the sender (human), implying that context-dependence is not a problem; and (b) Precisiation is carried out by the recipient (machine). In most applications of fuzzy logic, the precisiator is the sender (human). In most applications involving mechanization of natural language understanding, the precisiator is the recipient (machine). (a) is significantly simpler than (b).

Note. There is a close analogy between the concept of mm-precisiation and mathematical modeling. More specifically, the analog of mm-precisiation is mathematical modeling; the analog of precisand is the object of modeling; the analog of precisand is the model; and the analog of meaning is the input/output relation.

In the context of modeling, cointension is a measure of proximity of the input/output relations of the object of modeling and the model. A model is cointensive if its proximity is high.

The concept of cointension highlights an important issue. Specifically, what should be noted is that, in general, mm-precisiation of p is not the final objective. What matters is the cointension of an mm-precisand of p . In general, what are sought are mm-precisands which have high cointension. In other words, the desideratum is not merely mm-precisiation but, more importantly, cointensive mm-precisiation. As will be seen at a later point, a striking feature of fuzzy logic is its high power of cointensive precisiation.

The concept of cointensive mm-precisiation has an important implication for scientific theories. In large measure, scientific theories are based on bivalent logic. As a consequence, definitions of concepts are, generally, bivalent, in the sense that no degrees of truth are allowed. An example is the definition of bear market. The same applies to the definitions of recession, stability, independence, causality, stationarity, etc. The problem is that most of the concepts which are associated with bivalent definitions are in fact fuzzy, that is, are a matter of degree. For instance, the reason why the mm-precisand of bear market is not cointensive is rooted in the fact that bear market is a fuzzy concept. What can be said in a general way is that bivalent-logic-based definitions of fuzzy concepts cannot be expected to be cointensive, just as linear models of nonlinear systems cannot be expected to be good models. To achieve cointension what is needed is fuzzy logic.

Note. In the sequel, unless stated to the contrary, precisiation should be understood as cointensive mm-precisiation.

In the foregoing discussion, we talked about mm-precisiation but have not addressed a basic question: How can a proposition, p , be mm-precisiated? In fuzzy logic, a concept which plays a pivotal role in mm-precisiation is that of a generalized constraint. A brief discussion of this concept is presented in the following section.

4. The concept of a generalized constraint and the fundamental thesis of fuzzy logic

The concept of a constraint is high on the list of basic concepts in science. There is an extensive literature in the realms of mathematical programming and optimal control. Particularly worthy of note is the rapid growth of interest in constraint programming within computer science and related fields [58].

A basic assumption which is commonly made in the literature is that constraints are hard (inelastic) and are precisely defined. This assumption is not a good fit to reality. In most realistic settings, constraints have some elasticity and are not precisely defined. Familiar examples are:

Check-out time is 1 pm. A constraint on check-out time.

Speed limit is 100 km/h. A constraint on speed.

Vera has a son in mid-twenties and a daughter in mid-thirties. A constraint on Vera’s age.

How can such constraints be dealt with? A step was taken in my 1970 joint paper with Bellman “Decision making in a fuzzy environment” [4]. Another step was taken in my 1975 paper “Calculus of fuzzy restrictions” [82]. The concept of a generalized constraint goes far beyond; it was sketched in my 1986 paper “Outline of a computational approach to meaning and knowledge representation based on the concept of a generalized assignment statement” [95]. A more detailed description was given in my 1999 paper “From computing with numbers to computing with words – from manipulation of measurements to manipulation of perceptions” [99] and in my 2002 paper “Toward a perception-based theory of probabilistic reasoning with imprecise probabilities” [102] and in my 2005 and 2006 papers on the generalized theory of uncertainty (GTU) [104,107]. The concept of a generalized constraint plays a pivotal role in the nontraditional view of fuzzy logic which is the basis for the present paper.

The principal features of the concept of generalized constraint are summarized in Fig. 12 and et seq.

4.1. Constrained variable

The constraint variable, X , can assume a variety of forms. Among the principal forms are the following:

X is an n -ary variable, $X = (X_1, \dots, X_n)$

X is a proposition, e.g., Leslie is tall

X is a relation

X is a function of another variable: $X = f(Y)$

X is conditioned on another variable, X/Y

X is conditioned on another generalized constraint, X is r if Y is S

X has a structure, e.g., $X = \text{Location}(\text{Residence}(\text{Carol}))$

X is a generalized constraint, $X: Y$ is R

X is a group variable. In this case, there is a group, $G: (\text{Name}_1, \dots, \text{Name}_n)$, with each member of the group, Name_i , $i = 1, \dots, n$, associated with an attribute-value, h_i , of attribute H . h_i may be vector-valued. Symbolically

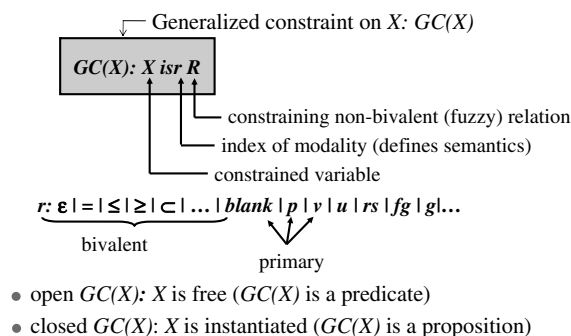


Fig. 12. Principal features of a generalized constraint.

$$G = (\text{Name}_1, \dots, \text{Name}_n)$$

$$G[H] = (\text{Name}_1/h_1, \dots, \text{Name}_n/h_n)$$

$$G[H \text{ is } A] = (\mu_A(h_1)/\text{Name}_1, \dots, \mu_A(h_n)/\text{Name}_n)$$

Basically, $G[H]$ is a relation and $G[H \text{ is } A]$ is a fuzzy restriction of $G[H]$ [81,82]

The concept of a group variable is closely related to the concept of a fuzzy relation

4.2. Modalities of generalized constraints

The indexical variable, r , defines the modality of a generalized constraint, that is, its semantics. The principal modalities are listed below.

$r: =$ equality constraint; $X = R$ is an abbreviation of $X \text{ is } = R$

$r: \leq$ inequality constraint; $X \leq R$

$r: \subset$ subsethood constraint; $X \subset R$

$r:$ blank possibilistic constraint; $X \text{ is } R$; R is the possibility distribution of X

$r: p$ probabilistic constraint; $X \text{ is } p R$; R is the probability distribution of X

$r: v$ veristic constraint; $X \text{ is } v R$; R is the verity (truth) distribution of X

$r: rs$ random set constraint; $X \text{ is } rs R$; R is the set-valued probability distribution of X

$r: fg$ fuzzy graph constraint; $X \text{ is } fg R$; X is a function and R is its fuzzy graph

$r: u$ usuality constraint; $X \text{ is } u R$ means usually ($X \text{ is } R$)

$r: g$ group constraint; $X \text{ is } g R$ means that R constrains the attribute-values of the group

$r: i$ iterated constraint; $X \text{ is } i R$ means that $X \text{ is } S$ and $S \text{ is } T$

To define the semantics of various modalities it is convenient to assume that the constrained variable, X , takes values in a finite set $U = (u_1, \dots, u_n)$. With this assumption, the semantics of various constraints may be defined as follows.

4.3. Possibilistic constraint

Consider the possibilistic constraint

$$X \text{ is } A$$

where A is a fuzzy set in U , defined as [81]

$$A = \mu_1/u_1 + \dots + \mu_n/u_n$$

with the understanding that $+$ is a separator and μ_i is the grade of membership of u_i in A , $i = 1, \dots, n$. The meaning of the possibilistic constraint, $X \text{ is } A$, is defined as

$$X \text{ is } A \xrightarrow{\text{definition}} \text{Poss}(X = u_i) = \mu_i, \quad i = 1, \dots, n.$$

4.4. Probabilistic constraint

Let P be a probability distribution defined on U . P may be expressed as [107]

$$P = p_1 \setminus u_1 + \dots + p_n \setminus u_n$$

In this case, $X \text{ is } p P \xrightarrow{\text{definition}} \text{Prob}(X = u_i) = p_i, \quad i = 1, \dots, n.$

It should be noted that p_i and u_i are allowed to take granular values, P_i and U_i , respectively, meaning that

$$p_i \text{ is } P_i \text{ and } u_i \text{ is } U_i, \quad i = 1, \dots, n$$

In this case, the probability distribution

$$P = P_1 \setminus U_1 + \dots + P_n \setminus U_n$$

is a granule-valued probability distribution in the sense defined in Section 2.1. Alternatively, a granule-valued distribution may be viewed as the result of granulation of the expression

$$P = p_1 \setminus u_1 + \dots + p_n \setminus u_n$$

A granular probability distribution may be defined as an iterated generalized constraint. More specifically

$$X \text{ is } P$$

$$P : \text{Prob}(X \text{ is } A_i) \text{ is } P_i, \quad i = 1, \dots, n$$

where the A_i are granules of X (Fig. 5). In this case, as in the case of granule-valued distributions, P is expressed as

$$P = P_1 \setminus U_1 + \dots + P_n \setminus U_n.$$

Note. Two examples will help to clarify the distinction between granular probability distributions and granule-valued probability distributions.

Example (a): Granular probability distribution. X is a real-valued random variable with probability distribution P . What is known about P is: Prob (X is small) is low; Prob (X is medium) is high; Prob (X is large) is low.

Example (b): Granule-valued probability distribution. X is a random variable taking the values small, medium and large with respective granular probabilities low, high and low. Question: What are the expected values of these probability distributions?

Note. The concepts of granular and granule-valued probability distributions are closely related to the concepts of granular and granule-valued possibility distributions.

4.5. Veristic constraint [99,104]

In this case, the semantics of $X \text{ is } A$ is defined by

$$X \text{ is } A \xrightarrow{\text{definition}} \text{Ver}(X = u_i) = \mu_i, \quad i = 1, \dots, n.$$

where $\text{Ver}(X = u_i)$ is the verity (truth) of the proposition $X = u_i$.

4.6. Fuzzy graph constraint

In this case, X is a function, f , from U to V . Assume that U and V are granulated, with the granules of U and V being A_1, \dots, A_m and B_1, \dots, B_n , respectively. The fuzzy graph, R , of f is defined as the disjunction of Cartesian products of the A_i and the $B_{j(i)}$ [80,84,96] (Fig. 13).

$$R = A_1 \times B_{j(1)} + \dots + A_m \times B_{j(m)}.$$

The fuzzy graph constraint is defined as the possibilistic constraint

$$f \text{ is } R \xrightarrow{\text{definition}} f \text{ is } (A_1 \times B_{j(1)} + \dots + A_m \times B_{j(m)}).$$

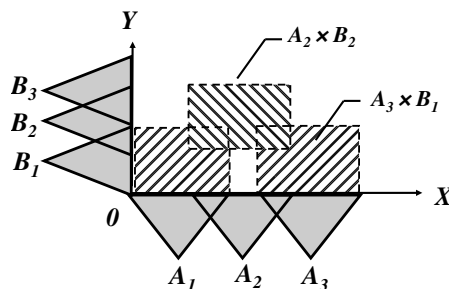


Fig. 13. Fuzzy graph.

4.7. Usuality constraint [99,102]

The constraint X is A is defined by

$$X \text{ is } A \xrightarrow{\text{definition}} \text{Prob}(X \text{ is } A) \text{ is usually,}$$

where usually is a fuzzy set in $[0, 1]$ which represents a fuzzy probability.

4.8. Primary constraints

Every conceivable constraint can be viewed as an instance of a generalized constraint. In practice, such generality is rarely needed. What is sufficient for most practical purposes is a subset of generalized constraints which can be generated from so-called primary constraints through combination, projection, qualification, propagation and counterpropagation. The primary constraints are: (a) possibilistic; (b) probabilistic; and (c) veristic. The primary constraints are somewhat analogous to the primary colors: red, green and blue.

4.9. Standard constraints

In large measure, scientific theories are based on what may be called standard constraints – constraints which are a subset of primary constraints. The standard constraints are: (a) bivalent possibilistic; (b) probabilistic; and (c) bivalent veristic.

What is important to note is that generality of generalized constraints goes far beyond the generality of standard constraints. What this points to is that the concept of a generalized constraint opens the door to a wide-ranging generalization of scientific theories. This aspect of fuzzy logic is discussed in greater detail in Sections 6 and 7.

4.10. Generalized constraint language (GCL) [99,104]

The concept of a generalized constraint serves as a basis for generation of what is referred to as the generalized constraint language (GCL). More specifically, GCL is generated by combination, projection, qualification, propagation and counterpropagation of generalized constraints. For example, combination of the possibilistic constraint

$$X \text{ isp } R$$

where X is a variable which takes values in a finite set, and the possibilistic constraint

$$(X, Y) \text{ is } S$$

generates the fuzzy random set constraint [104,107]

$$Y \text{ isfrs } T$$

Fuzzy random sets are closely related to fuzzy-set-valued random variables. There is an extensive literature on fuzzy-set-valued random variables [52,62,25,46,55,9,51]. Random sets and set-valued random variables are closely related to the Dempster–Shafer theory of evidence [11,60,59]. An extension of the Dempster–Shafer theory to fuzzy sets and fuzzy probabilities is sketched in [87,89].

GCL is an open language in the sense that generalized constraints may be added to GCL at will. Simple examples of generalized constraints in GCL are

$$(X \text{ is } R) \text{ and } ((X, Y) \text{ isp } S)$$

$$(X \text{ isu } R) \text{ and } (Y \text{ isu } S)$$

$$(X \text{ is } R) \text{ isp } S$$

$$(Y \text{ isu } B) \text{ if } (X \text{ is } A)$$

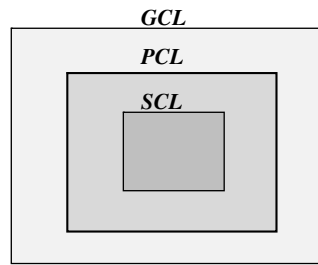


Fig. 14. GCL, PCL, and SCL.

In relation to GCL, PCL (primary constraint language) and SCL (standard constraint language) are subsets of GCL which are generated, respectively, by primary and standard constraints (Fig. 14).

Note. GCL is more than a language – it is a language system. A language has descriptive capability. A language system has deductive capability in addition to descriptive capability. GCL has both capabilities.

The concept of a generalized constraint plays a pivotal role in fuzzy logic. In particular, it serves to precisiate the concepts of information and meaning. More specifically, the fundamental thesis of fuzzy logic is that information may be represented as a generalized constraint.

information = generalized constraint

with the understanding that information relates to the value of a variable, X , to which the generalized constraint applies. For example, if the information about X is that X is small, then this information may be represented as a possibilistic constraint

X is small

with small being a granular value of X . It should be noted that the traditional view that information is statistical in nature is a special, albeit important case of viewing information as a generalized constraint. Another point which should be noted is that in information theory the primary concern is not with the substance of information but with its measure. The fundamental thesis relates to substance rather than measure [16,35].

An important corollary of the fundamental thesis is the meaning postulate of fuzzy logic. More specifically, let p be a proposition. A proposition is a carrier of information. Consequently,

meaning of p = generalized constraint.

More concretely,

meaning of proposition = closed generalized constraint

meaning of predicate = open generalized constraint.

The meaning postulate leads to an important connection between the concept of a generalized constraint and the concept of mm-precisiation. More specifically, what can be concluded is the equality

mm-precisiation = generalized constraint.

Equivalently,

mm-precisiation = translation into GCL.

This equality may be viewed as a more concrete statement of the meaning postulate. Transparency of translation may be enhanced through annotation. Details may be found in Section 7 [107].

The meaning postulate points to an important aspect of translation into GCL.

Let SCL denote the subset of GCL which is associated with standard constraints, that is, bivalent possibilistic, probabilistic and bivalent veristic constraints. Let p be a proposition. An mm-precisiand of p , p^* , is an element of GCL.

Let $C(p^*, p)$ be the cointension of p^* in relation to p , and let $\sup_{\text{GCL}} C(p^*, p)$ and $\sup_{\text{SCL}} C(p^*, p)$ be the suprema of $C(p^*, p)$ over GCL and SCL, respectively. Since SCL is a subset of GCL we have

$$\sup_{\text{GCL}} C(p^*, p) \geq \sup_{\text{SCL}} C(p^*, p)$$

This obvious inequality has an important implication. Specifically, as a meaning precisiation language, fuzzy logic dominates bivalent logic. As a very simple example consider the proposition p : Speed limit is 65 mph. Realistically, what is the meaning of p ? The inequality implies that employment of fuzzy logic for precisiation of p would lead to a precisiand whose cointension is at least as high – and generally significantly higher – than the cointension which is achievable through the use of bivalent logic.

In addition to serving as a meaning precisiation language, GCL serves another important function – the function of a deductive question-answering system. In this role, what matters are the rules of deduction. In GCL, the rules of deduction coincide with the rules which govern constraint propagation and counterpropagation. Basically, these are the rules which govern generation of a generalized constraint from other generalized constraints [107].

The principal rule of deduction in fuzzy logic is the so-called extension principle [81]. The extension principle can assume a variety of forms, depending on the generalized constraints to which it applies. A basic form which involves possibilistic constraints is the following. An analogous principle applies to probabilistic constraints.

Let X be a variable which takes values in U , and let f be a function from U to V . The point of departure is a possibilistic constraint on $f(X)$ expressed as

$$f(X) \text{ is } A$$

where A is a fuzzy relation in V which is defined by its membership function $\mu_A(v)$, $v \in V$.

Let g be a function from U to W . The possibilistic constraint on $f(X)$ induces a possibilistic constraint on $g(X)$ which may be expressed as

$$g(X) \text{ is } B$$

where B is a fuzzy relation. The question is: What is B ?

The extension principle reduces the problem of computation of B to the solution of a variational problem. Specifically

$$\frac{f(X) \text{ is } A}{g(X) \text{ is } B}$$

where $\mu_B(w) = \sup_u \mu_A(f(u))$

subject to

$$w = g(u)$$

The structure of the solution is depicted in Fig. 15. Basically, the possibilistic constraint on $f(X)$ counterpropagates to a possibilistic constraint on X . Then, the possibilistic constraint on X propagates to a possibilistic constraint on $g(X)$.

There is a version of the extension principle – referred to as the fuzzy graph extension principle – which plays an important role in control and systems analysis [84,96]. More specifically, let f be a function from reals to reals, $Y = f(X)$. Let $*f$ and $*X$ be the granulands of f and X , respectively, with $*f$ having the form of a fuzzy graph (Section 2.1).

$$*f = A_1 \times B_{j(1)} + \dots + A_m \times B_{j(m)}$$

where the A_i , $i = 1, \dots, m$ and the B_j , $B = 1, \dots, n$ are granules of X and Y , respectively; \times denotes Cartesian product; and $+$ denotes disjunction (Fig. 16).

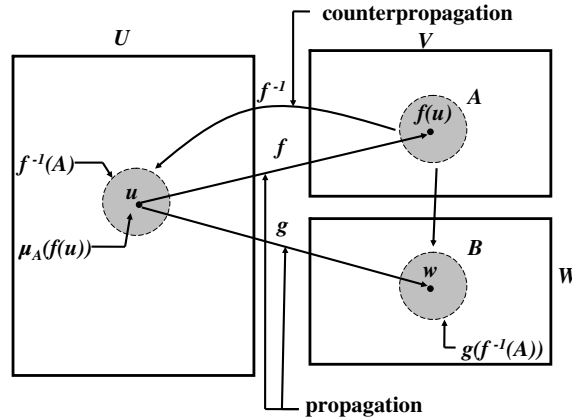


Fig. 15. Structure of the extension principle.

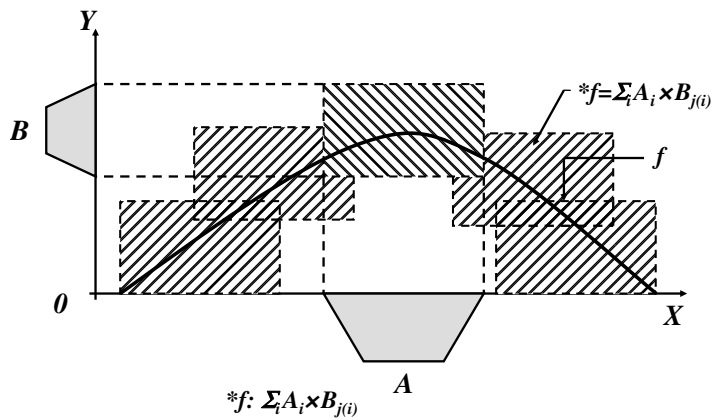


Fig. 16. Fuzzy graph extension principle. $B = {}^*f(A)$.

In this instance, the extension principle may be expressed as follows:

X is A

$$f \text{ is } (A_1 \times B_{j(1)} + \dots + A_m \times B_{j(m)})$$

$$Y \text{ is } (m_1 \wedge B_{j(1)} + \dots + m_m \wedge B_{j(m)})$$

where the m_i are matching coefficients, defined as [84]

$$m_i = \sup(A \cap A_i), \quad i = 1, \dots, m$$

and \wedge denotes conjunction (min). In the special case where X is a number, a , the possibilistic constraint on Y may be expressed as

$$Y \text{ is } (\mu_{A_1}(a) \wedge B_{j(1)} + \dots + \mu_{A_m}(a) \wedge B_{j(m)})$$

In this form, the extension principle plays a key role in the Mamdani–Assilian fuzzy logic controller [42].

4.11. Deduction

Assume that we are given an information set, I , which consists of a system of propositions (p_1, \dots, p_n) . I will be referred to as the initial information set. The canonical problem of deductive question-answering is that of computing an answer to q , $\text{ans}(q|I)$, given I [106,107].

The first step is to ask the question: What information is needed to answer q ? Suppose that the needed information consists of the values of the variables X_1, \dots, X_n . Thus

$$\text{ans}(q|I) = g(X_1, \dots, X_n)$$

where g is a known function.

Using GCL as a meaning precisiation language, the initial information set may be expressed as a generalized constraint on X_1, \dots, X_n . In the special case of possibilistic constraints, the constraint on the X_i may be expressed as

$$f(X_1, \dots, X_n) \text{ is } A$$

where A is a fuzzy relation.

At this point, what we have is (a) a possibilistic constraint induced by the initial information set, and (b) an answer to q expressed as

$$\text{ans}(q|I) = g(X_1, \dots, X_n),$$

with the understanding that the possibilistic constraint on f propagates to a possibilistic constraint on g . To compute the induced constraint on g what is needed is the extension principle of fuzzy logic [81].

As a simple illustration of deduction, it is convenient to use an example which was considered earlier.

Initial information set, p : Most Swedes are tall.

Question, q : What is the average height of Swedes?

What information is needed to compute the answer to q ? Let P be a population of n Swedes, Swede₁, ..., Swede _{n} . Let h_i be the height of Swede _{i} , $i = 1, \dots, n$. Knowing the h_i , the answer to q can be expressed as

$$\text{av}(h) : \text{ans}(q|p) = \frac{1}{n}(h_1 + \dots + h_n)$$

Turning to the constraint induced by p we note that the mm-precisiand of p may be expressed as the possibilistic constraint

$$p \xrightarrow{\text{mm-precisiation}} \frac{1}{n} \text{Count}(\text{tall.Swedes}) \text{ is most}$$

where $\text{Count}(\text{tall.Swedes})$ is the number of tall.Swedes in P , with the understanding that tall.Swedes is a fuzzy subset of P . In fuzzy logic, a simple version of the number of elements in a fuzzy set, A ,

$$A = (\mu_A(u_1)/u_1, \dots, \mu_A(u_n)/u_n)$$

is defined as

$$\Sigma \text{Count}(A) = (\mu_A(u_1) + \dots + \mu_A(u_n))$$

Using this definition of ΣCount [91,92], the expression for the constraint on the h_i may be written as

$$\frac{1}{n}(\mu_A(u_1) + \dots + \mu_A(u_n)) \text{ is most}$$

At this point, application of the extension principle leads to a solution which may be expressed as

$$(\mu_{\text{av}(h)}(v)) = \text{sup}_h \left(\frac{1}{n}(\mu_{\text{most}} \mu_{\text{tall}}(h_1) + \dots + \mu_{\text{tall}}(h_n)) \right), \quad h = (h_1, \dots, h_n)$$

subject to

$$v = \frac{1}{n}(h_1 + \dots + h_n)$$

In summary, the generalized constraint language is, by construction, maximally expressive. Importantly, what this implies is that, in realistic settings, fuzzy logic, viewed as a modeling language, has a significantly higher level of power and generality than modeling languages based on standard constraints or, equivalently, on bivalent logic and bivalent-logic-based probability theory.

5. Part 2 What does fuzzy logic have to offer?

The most visible, the best understood and the most widely used contribution of fuzzy logic is the concept of a linguistic variable and the associated machinery of fuzzy if–then rules. But there are other equally important contributions which are much less visible and much less well understood. What is needed to understand the significance of these contributions is fuzzy logic in its nontraditional setting.

A concise discussion of the basic concepts and ideas which underlie some of the more important of these contributions is presented in Part 2. For convenience, discussions of these contributions are preceded by a synopsis of what fuzzy logic has to offer.

- Linguistic variables and fuzzy if–then rules

The machinery of linguistic variables and fuzzy if–then rules is unique to fuzzy logic. This machinery has played and is continuing to play a pivotal role in the conception and design of control systems and consumer products. However, its applicability is much broader. A key idea which underlies the machinery of linguistic variables and fuzzy if–then rules is centered on the use of information compression. In fuzzy logic, information compression is achieved through the use of fuzzy granulation.

- FL-generalization

Fuzzy logic has far greater generality than bivalent logic. This implies that any bivalent-logic-based theory, T , may be generalized – and hence upgraded – through what is referred to as FL-generalization. FL-generalization of a bivalent-logic-based theory, T , involves addition to T of concepts and techniques drawn from fuzzy logic. In the limit, FL-generalization of T leads to a fuzzy-logic-based theory, T^+ . By construction, T^+ has a much higher level of generality than T , and hence has enhanced capability to deal with imprecision, uncertainty, incompleteness of information, partiality of truth and partiality of possibility.

- The concepts of precisiation and cointension

The concepts of precisiation and cointension play important roles in nontraditional view of fuzzy logic. In nontraditional fuzzy logic, differentiation is made between two concepts of precision: precision of value, v-precision; and precision of meaning, m-precision. Furthermore, differentiation is made between precisiation of meaning which is (a) human-oriented, or mh-precisiation for short; and (b) machine-oriented, or mm-precisiation for short. It is understood that mm-precisiation is mathematically well defined. The object of precisiation, p , and the result of precisiation, p^* , are referred to as precisand and precisand, respectively. Informally, cointension is defined as a measure of closeness of the meanings of p and p^* . Precisiation is cointensive if the meaning of p^* is close to the meaning of p . One of the important features of fuzzy logic is its high power of cointensive precisiation. What this implies is that better models of reality can be achieved through the use of fuzzy logic.

Cointensive precisiation has an important implication for science. In large measure, science is bivalent-logic-based. In consequence, in science it is traditional to define concepts in a bivalent framework, with no degrees of truth allowed. The problem is that, in reality, many concepts in science are fuzzy, that is, are a matter of degree. For this reason, bivalent-logic-based definitions of scientific concepts are, in many cases, not cointensive. To formulate cointensive definitions of fuzzy concepts it is necessary to employ fuzzy logic.

- NL-Computation, computing with words (CW) and precisiated natural language (PNL)

Much of human knowledge is expressed in natural language. Traditional theories of natural language are based on bivalent logic. The problem is that natural languages are intrinsically imprecise. Imprecision of natural languages is rooted in imprecision of perceptions. A natural language is basically a system for describing perceptions. Perceptions are intrinsically imprecise, reflecting the bounded ability of human sensory organs, and ultimately the brain, to resolve detail and store information. Imprecision of perceptions is passed on to natural languages.

Bivalent logic is intolerant of imprecision, partiality of truth and partiality of possibility. For this reason, bivalent logic is intrinsically unsuited to serve as a foundation for theories of natural language. As the logic of imprecision and approximate reasoning, fuzzy logic is a much better choice.

NL-Computation, computing with words (CW) and precisiated natural language (PNL) are closely related formalisms [99–101,103]. In conventional modes of computation, the objects of computation

are mathematical constructs. By contrast, in NL-Computation the objects of computation are propositions and predicates drawn from a natural language. A key idea which underlies NL-Computation involves representing the meaning of propositions and predicates as generalized constraints. NL-Computation opens the door to a wide-ranging enlargement of the role of natural languages in scientific theories [32,33,65,39,61,64].

- Computational theory of perceptions

Humans have a remarkable capability to perform a wide variety of physical and mental tasks without any measurements and any computations. In performing such tasks humans employ perceptions. To endow machines with this capability what is needed is a formalism in which perceptions can play the role of objects of computation. The fuzzy-logic-based computational theory of perceptions serves this purpose. A key idea in this theory is that of computing not with perceptions per se, but with their descriptions in a natural language. Representing perceptions as propositions drawn from a natural language opens the door to application of NL-Computation to computation with perceptions. Computational theory of perceptions is of direct relevance to achievement of human level machine intelligence.

- Possibility theory

Possibility theory is a branch of fuzzy logic. Possibility theory and probability theory are distinct theories. Possibility theory may be viewed as a formalization of perception of possibility, whereas probability theory is rooted in perception of likelihood. In large measure, possibility theory and probability theory are complementary rather than competitive. Possibility theory is of direct relevance to, knowledge representation, semantics of natural languages, decision analysis and computation with imprecise probabilities.

- Computation with imprecise probabilities

Most real-world probabilities are perceptions of likelihood. As such, real-world probabilities are intrinsically imprecise. Until recently, the issue of imprecise probabilities has been accorded little attention in the literature of probability theory. More recently, the problem of computation with imprecise probabilities has been an object of rapidly growing interest [67].

Typically, imprecise probabilities occur in an environment of imprecisely defined variables, functions, relations, events, etc. Existing approaches to computation with imprecise probabilities do not address this reality. To address this reality what is needed is fuzzy logic and, more particularly, NL-Computation and the computational theory of perceptions. A step in this direction was taken in my 2002 paper “Toward a perception-based theory of probabilistic reasoning with imprecise probabilities;” in my 2005 paper “Toward a generalized theory of uncertainty (GTU) – an outline,” and my 2006 paper “Generalized theory of uncertainty (GTU)–principal concepts and ideas” [102,104,107].

- Fuzzy logic as a modeling language

Science deals not with reality but with models of reality. More often than not, reality is fuzzy. For this reason, construction of realistic models of reality calls for the use of fuzzy logic rather than bivalent logic.

Fuzzy logic is a logic of imprecision and approximate reasoning. It is natural to employ fuzzy logic as a modeling language when the objects of modeling are not well defined. But what is somewhat paradoxical is that in many of its practical applications fuzzy logic is used as a modeling language for systems which are precisely defined. The explanation is that, in general, precision carries a cost. In those cases in which there is a tolerance for imprecision, reduction in cost may be achieved through imprecisiation, e.g., data compression, information compression and summarization. The result of imprecisiation is an object of modeling which is not precisely defined. A fuzzy modeling language comes into play at this point. This is the key idea which underlies the fuzzy logic gambit. The fuzzy logic gambit is widely used in the design of consumer products – a realm in which cost is an important consideration.

6. Fuzzy logic as the basis for generalization of scientific theories

By construction, fuzzy logic has a much more general conceptual structure than bivalent logic. A key concept in the transition from bivalent logic to fuzzy logic is the generalization of the concept of a set to a fuzzy set. This generalization is the point of departure for what will be referred to as FL-generalization.

More specifically, FL-generalization of any theory, T , involves an addition to T of concepts drawn from fuzzy logic. In the limit, as more and more concepts which are drawn from fuzzy logic are added to T , the foundation of T is shifted from bivalent logic to fuzzy logic. By construction, FL-generalization results in an upgraded theory, T^+ , which is at least as rich and, in general, significantly richer than T .

As an illustration, consider probability theory, PT – a theory which is bivalent-logic-based. Among the basic concepts drawn from fuzzy logic which may be added to PT are the following [102]:

set + fuzzy set
 event + fuzzy event
 relation + fuzzy relation
 probability + fuzzy probability
 random set + fuzzy random set
 independence + fuzzy independence
 stationarity + fuzzy stationarity
 random variable + fuzzy random variable
 etc.

As a theory, PT^+ is much richer than PT. In particular, it provides a basis for construction of models which are much closer to reality than those that can be constructed through the use of PT. This applies, in particular, to computation with imprecise probabilities.

Some scientific theories have already been FL-generalized to some degree, and many more are likely to be FL-generalized in coming years. Particularly worthy of note are the following FL-generalizations.

control → fuzzy control [12,20]
 linear programming → fuzzy linear programming [108,21]
 probability theory → fuzzy probability theory [102,104,107]
 measure theory → fuzzy measure theory [14,66]
 topology → fuzzy topology [73,41]
 graph theory → fuzzy graph theory [36,44]
 cluster analysis → fuzzy cluster analysis [6,29]
 Prolog → fuzzy Prolog [45,23]
 etc.

FL-generalization is a basis for an important rationale for the use of fuzzy logic. It is conceivable that eventually the foundations of many scientific theories may be shifted from bivalent logic to fuzzy logic.

7. Fuzzy logic and natural language

Much of human knowledge is described in natural language. Furthermore, natural languages have a position of centrality in human reasoning and communication. For this reasons, as we move further into the age of machine intelligence and automated decision-making, the problem of mechanization of natural language understanding is certain to grow in visibility and importance. Natural languages are pervasively imprecise in the sense that in a natural language almost everything is a matter of degree. Imprecision of natural languages is rooted in imprecision of perceptions. Basically, a natural language is a system for describing perceptions. Perceptions are intrinsically imprecise, reflecting the bounded ability of human sensory organs, and ultimately the brain, to resolve detail and store information [48]. Imprecision of perceptions is passed on to natural languages. This is the principal reason why natural languages are pervasively imprecise. Imprecision of natural languages is an issue of central importance. What is remarkable is that despite its importance, the issue of imprecision has been and continues to be largely ignored in the literatures of linguistics and philosophy of languages. There is an explanation. In large measure, theories of natural language are based on bivalent logic. Bivalent logic is intolerant of imprecision and partial truth. This is why bivalent-logic-based theories of natural language are intrinsically incapable of coming to grips with

the issue of imprecision. What is widely unrecognized is that to close the wide gap between the precision of bivalent logic and the imprecision of natural languages what is needed is fuzzy logic, in addition to probability theory. This is the conclusion that I arrived at in the course of many years of exploration of ways in which fuzzy logic can be applied to the development of a better understanding of how to deal with imprecision of natural languages. It should be noted that in fuzzy logic, as in natural languages, everything is or is allowed to be a matter of degree.

A bit of history. My exploration of application of fuzzy logic to natural languages began within a few years of the publication of my first paper on fuzzy sets. My first paper, entitled “Quantitative fuzzy semantics”, was published in 1971 [76]. My second paper entitled “A fuzzy-set-theoretic interpretation of linguistic hedges”, was published in 1972 [77,78]. Many others followed. Particularly worthy of note is my 1978 paper “PRUF—a meaning representation language for natural languages”, in which the concept of precisiation was introduced [86,93]. My 1982 paper “Test-score semantics for natural languages and meaning representation via PRUF”, was the first in a series of papers on test-score semantics [90], followed by my 1983 paper “Test-score semantics as a basis for a computational approach to the representation of meaning” [91,94]. My 1983 paper “A computational approach to fuzzy quantifiers in natural languages”, dealt with fuzzy quantifiers [92]; and my 1986 paper “Outline of a computational approach to meaning and knowledge representation based on the concept of a generalized assignment statement”, introduced the concept of a generalized constraint [95]. The concept of a generalized constraint opened the door to the development of granular computing, rooted in my 1997 paper “Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic” [97]. The formalisms of computing with words and perceptions, and NL-Computation were introduced in my 1999 paper “From computing with numbers to computing with words – from manipulation of measurements to manipulation of perceptions” [99]; and developed further in “A new direction in AI – toward a computational theory of perceptions” [101]; “Precisiated natural language (PNL)” [103]; and “The generalized theory of uncertainty (GTU) – principal concepts and ideas” [107]. A key idea which was developed in these and related papers is that there is a naturally close relationship between the concepts of knowledge, meaning and constraint.

The constraint-oriented approach to the semantics of natural languages which was the principal theme of my papers reflected my background in systems analysis and optimization. My ideas were too far removed from the mainstream to find resonance within the linguistics and philosophy of languages communities. Within the fuzzy logic community, there was and still is a lack of interest in application of fuzzy logic to natural languages. A notable exception is the mathematically rigorous work of Novak [50].

The relation between fuzzy logic and natural languages is two-sided. On one side, fuzzy logic draws on natural languages in the formalism of the linguistic variable and the calculi of fuzzy if-then rules. On the other side, as the logic of imprecision and approximate reasoning, fuzzy logic has a great deal to contribute to formalization of natural languages, especially in the realm of semantics, and may well play an essential role in mechanization of natural language understanding.

In my papers on fuzzy logic and natural languages, attention is focused on semantics, knowledge representation and deduction. More specifically, two approaches to semantics are described: test-score semantics (TSS) [90,91,94] and generalized-constraint-based semantics (GCS) [107]. Test-score semantics is in the spirit of truth-conditional and possible-world semantics [38,10]. However, in the context of fuzzy logic rather than bivalent logic, generalized constraint-based semantics has a significantly higher level of generality than truth conditional semantics and possible-world semantics. Generalized-constraint-based semantics is simpler than test-score semantics and has lesser generality. In my view, the most natural and easiest to understand semantics of natural languages is GCS. A brief exposition of GCS is presented in the following.

For simplicity, attention will be focused on the semantics of propositions, with the understanding that similar results apply to predicates, questions, commands and other semantic entities. The point of departure in GCS is the fundamental thesis of fuzzy logic, namely

information = generalized constraint

A consequence of the fundamental thesis is the meaning postulate.

meaning of p = generalized constraint

In NL-Computation, the meaning of p is equated to its mm-precisiand. More specifically,

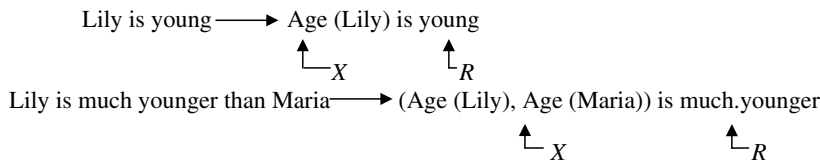
$$\text{proposition} \longrightarrow \text{mm-precisation} \longrightarrow X \text{ isr } R$$

The rationale for mm-precisation is the following. A proposition, p , may be viewed as an answer to a question, q : What is the value of X ? where X is explicit or implicit in p . Likewise, a generalized constraint may be viewed as an answer to a question: What is the value of X ? Consequently, the meaning of p may be expressed as a generalized constraint.

$$X \text{ isr } R$$

Note that X is a variable that is focused on but is not uniquely determined by p . For this reason, X is referred to as a focal variable.

Simple examples (possibilistic constraints).



In these simple examples X and R can be identified by inspection. More generally, X and R are defined by procedures which act on a relational database termed Explanatory Database or ED for short, with ED serving the purpose of precisiating the meaning of p [90,91]. Informally, ED may be viewed as the information which is needed to compute X and R . More concretely, ED may be viewed as a description of a possible world [38,10]. In general, construction of ED requires world knowledge. As an illustration, consider an example which was used earlier in Section 4, p : Most Swedes are tall. For this example, the ED may be expressed as

$$\begin{aligned}
 \text{ED} &= \text{POPULATION.SWEDES}[\text{Name}; \text{Height}] + \\
 &\text{TALL} [\text{Height}; \mu\text{Tall}] + \\
 &\text{MOST} [\text{Proportion}; \mu\text{Most}]
 \end{aligned}$$

In ED, POLULATION.SWEDES is a relation with attributes Name, Height; TALL is a fuzzy relation in which μTall is the grade of membership of a value of Height in TALL; and MOST is a fuzzy relation in which μMost is the grade of membership of a value of Proportion in MOST. The symbol $+$ serves as a separator.

Note. TALL is an intensional predicate in the sense that its meaning is attribute-based

Note. Database variables are the attribute values in POPULATION.SWEDES. In the example under consideration, the database variables are h_1, \dots, h_n , the values of Height.

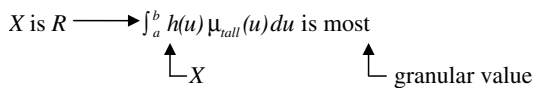
In the example, X is equated to the fraction of tall Swedes among Swedes, and R is equated to most. Alternatively, X may be defined in terms of the height density function, $h(u)$, where

$$h(u) du = \text{fraction of Swedes whose height is in } [u, u + du], \quad a \leq u \leq b$$

$$\int_a^b h(u) du = 1$$

$$\text{fraction of tall Swedes} = \int_a^b h(u) \mu_{\text{tall}}(u) du$$

Consequently,



which is a generalized constraint of the form (Section 4)

$$f(X) \text{ is } R$$

What should be stressed is that precisiation of meaning is not the ultimate objective. The real objective is deductive question-answering, given an information set, I , which consists of a system of propositions (p_1, \dots, p_n) . In the context of this objective, precisiation of the meanings of I and q is a first step in the deduction/computation process.

As was noted earlier, in fuzzy logic the deduction/computation process involves a collection of rules which govern generation of generalized constraints. The principal deduction/computation rule is the extension principle [74,81] (Section 4). Applying this principle to the problem under consideration, we have

$$\begin{aligned} p: & \text{Most Swedes are tall} \\ q: & \text{Fraction of short Swedes?} \\ & \int_a^b h(u)\mu_{\text{tall}}(u) du \text{ is most} \leftarrow \text{given} \\ & \int_a^b h(u)\mu_{\text{short}}(u) du \text{ is } q \leftarrow \text{needed} \end{aligned}$$

Using the extension principle, the solution may be expressed as

$$\mu_q(v) = \sup_u \left(\mu_{\text{most}} \left(\int_a^b h(u)\mu_{\text{tall}}(u) du \right) \right)$$

subject to:

$$\begin{aligned} v &= \int_a^b h(u)\mu_{\text{short}}(u) du \\ \int_a^b h(u) du &= 1 \end{aligned}$$

In summary, through the use of precisiation and deduction/computation fuzzy logic opens the door to deduction/computation with information described in natural language. This is the core of NL-Computation. NL-Computation is one of the principal contributions of fuzzy logic. In particular, NL-Computation is of direct relevance to mechanization of natural language understanding, as well as to dealing with problems in which there is uncertainty about uncertainty, or uncertainty² for short. The problem of how to deal with uncertainty² is certain to grow in visibility and importance in coming years.

8. Fuzzy logic as a modeling language

In large measure, science deals not with reality but with models of reality. In this perspective, scientific progress is driven, in large measure, by a quest for better models of reality.

Let $M(S)$ be a model of a system, S . In the context of modeling, the cointension of $M(S)$ is a measure of the closeness of input/output relations of S and $M(S)$. Generally, $M(S)$ is described in a mathematical language. In this sense, various branches of mathematics such as probability theory, differential equations, difference equations, functional analysis, etc., may be viewed as modeling languages. Generally, such modeling languages are based on bivalent logic. Bivalent logic is a special case of fuzzy logic. In consequence, fuzzy-logic-based modeling languages have, in general, higher power of cointension than their bivalent-logic-based counterparts. This does not mean, however, that a fuzzy-logic-based model is as good or better than a bivalent-logic-based model because computational complexity may be an important consideration. Informally, what can be asserted is that given the most powerful bivalent-logic-based modeling language there exists a fuzzy-logic-based modeling language which dominates it. In a related way, any bivalent-logic-based modeling language can be FL-generalized and hence, upgraded, through addition of concepts and techniques drawn from fuzzy logic. An example of a widely used fuzzy-logic-based modeling language is the language of fuzzy if-then rules. Another example is the fuzzy-logic-based language of the generalized theory of uncertainty (GTU) – a language which has the potential for playing an essential role in cointensive modeling of systems such as economic systems, social systems, political systems, etc. – systems in which what is needed is a machinery for computation with imprecise probabilities, imprecise goals and imprecise constraints.

Taking a more general view, let us say that a system, S , is perfect information based, or a p-system for short, if in the description of S there is no imprecision, no uncertainty, no incompleteness of information, no partiality of truth and no partiality of possibility. Correspondingly, S is an imperfect information based system, or ip-system for short, if it is not a p-system.

If S is an ip-system, then by construction, higher or equally high level of cointension can be achieved with a fuzzy-logic-based modeling language than with its bivalent-logic-based counterparts. To understand the argument, an analogy may be helpful. Specifically, the class of linear systems is a subclass of nonlinear systems. Consequently, if S is a nonlinear system, then a higher or equally high degree of cointension can be achieved through the use of a nonlinear modeling language than through the use of a linear modeling language. This conclusion serves as a basis for an important rationale for the use of fuzzy logic. Specifically,

Rationale A. For imperfect information systems (ip-systems) fuzzy logic is the logic of choice as a basis for modeling languages.

Rationale A asserts that fuzzy logic is the logic of choice for modeling of ip-systems. This is to be expected since fuzzy logic is much more general and has a much wider scope than bivalent logic. But what would not be expected – and indeed seems to be somewhat paradoxical – is that today most of the applications of fuzzy logic relate to p-systems. How can this be explained?

The explanation is rooted in my 1973 paper “Outline of a new approach to the analysis of complex systems and decision processes”, in which the concepts of a linguistic variable and fuzzy if–then rules are introduced [79]. It is this paper that opened the door to applications of fuzzy logic to p-systems and played a pivotal role in the later development of computing with words and NL-Computation.

The key idea in my 1973 paper is simple but not obvious. Let S be a p-system, e.g., an electrical network with precisely known components. Normally, a fuzzy-logic-based modeling language would not be used to construct a model of a p-system. But there is an important exception. Precision carries a cost. If in using S there is a tolerance for imprecision, then this tolerance might be exploited to reduce cost. Familiar examples are data compression and summarization. What fuzzy logic has to offer is a much more general approach to exploitation of a tolerance for imprecision. Specifically, through the use of fuzzy logic, S is deliberately v-impresiated, resulting in an ip-system, $*S$. Then, a fuzzy-logic-based modeling language may be employed to construct a cointensive model of $*S$. Because a deliberate sacrifice in precision is involved, the artifice is referred to as the fuzzy logic gambit. Basically, the fuzzy logic gambit involves deliberate v-impresiation followed by mm-precision. The fuzzy logic gambit is the basis for Rationale B.

Rationale B. Let S be a p-system. If S is a p-system which has a degree of tolerance for imprecision, then fuzzy logic may be employed to exploit the tolerance for imprecision by a deliberate use of v-impresiation to convert S into an ip-system $*S$. Then a fuzzy-logic-based modeling language may be used to construct a cointensive model of $*S$.

As a very simple example assume that I have precise knowledge of Lily’s birthdate – the year, the day and the time. Suppose that I am asked “How old is Lily?” Knowing that precise information is not needed, I respond by saying “Lily is young”. Then I precisiate “young” by defining “young” as a fuzzy set. This example is a very simple instance of the fuzzy logic gambit.

A more substantive example is Professor Yamakawa’s use of fuzzy logic to stabilize an inverted pendulum [69] (Fig. 17). Using the traditional approach, the first step is that of formulating a differential-equation-based model of S . Then, methods drawn from the theory of stability may be used to stabilize S .

In Yamakawa’s approach the point of departure is a fuzzy if–then rules-based model of S involving a stabilizing input, y . What Professor Yamakawa showed is that only seven simple if–then rules are needed to stabilize the pendulum. Specifically,

IF θ is PM AND $\dot{\theta}$ is ZR, THEN \dot{y} is PM,
 IF θ is PS AND $\dot{\theta}$ is PS, THEN \dot{y} is PS,
 IF θ is PS AND $\dot{\theta}$ is NS, THEN \dot{y} is ZR,
 IF θ is NM AND $\dot{\theta}$ is ZR, THEN \dot{y} is NM,
 IF θ is NS AND $\dot{\theta}$ is NS, THEN \dot{y} is NS,
 IF θ is ZR AND $\dot{\theta}$ is ZR, THEN \dot{y} is ZR.

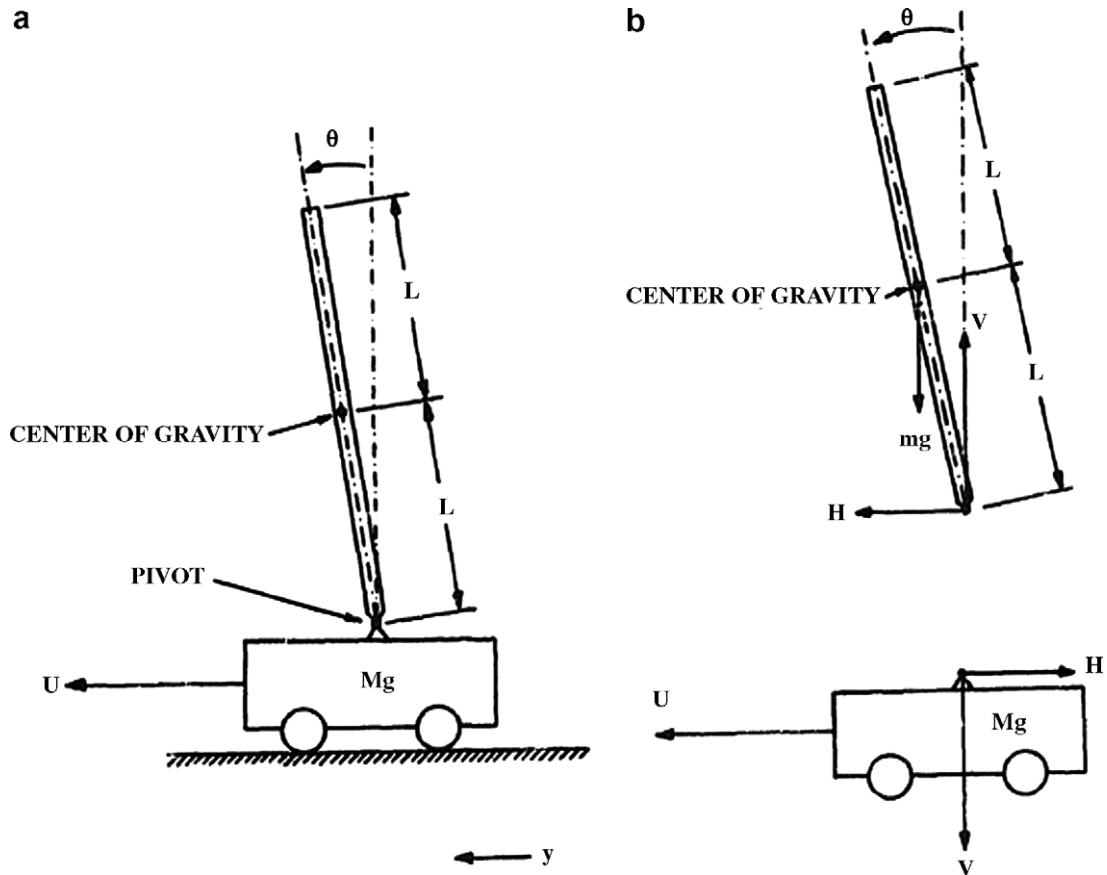


Fig. 17. Yamakawa's inverted pendulum.

In these rules, NS and PS denote Negative Small and Positive Small, respectively; NM and PM denote Negative Medium and Positive Medium, respectively; NL and PL denote Negative Large and Positive Large, respectively; ZR denotes zero; and y is the stabilizing input. Such rules can be formulated without any knowledge of dynamics or fuzzy logic. It is this remarkable ability of fuzzy if-then rules modeling language to bypass the need for bivalent-logic-based models that explains why fuzzy logic is employed so widely in applications ranging from industrial control to consumer products.

8.1. Concluding remarks

Harsh criticisms of fuzzy logic reflect misconceptions about what it is and what it has to offer. To begin with, fuzzy logic is not fuzzy. Basically, fuzzy logic is a precise logic of imprecision and approximate reasoning. The real-world is pervaded with fuzziness. Fuzzy logic is needed to deal effectively with fuzzy reality.

An important point that has to be made is that fuzzy logic is much more than a logical system. Fuzzy logic has many facets. Mathematically, the logical facet and the fuzzy-set-theoretic facet are the basic facets of fuzzy logic. A facet which plays a pivotal role in almost all applications of fuzzy logic is the relational facet – a facet which is focused on linguistic variables and fuzzy if-then rules.

In this paper, fuzzy logic is viewed in a nontraditional perspective. In this view, the concept of precision is elevated to the status of a cornerstone of fuzzy logic. This status reflects what is not widely recognized, namely, that one of the most important contributions of fuzzy logic is its high power of precision. It is this power that underlies the ability of fuzzy logic to serve as a coextensive model of reality, especially in human-centric fields such as economics, law, linguistics and psychology.

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